Here’s how these problem-solving sessions work.

Start by working on the problem indicated by the number on the front of this sheet. If you find a solution, let me know and sketch it for me. If it’s correct, you then present it on the board to the class. Next, you write up your solution within one week, in a format I can post for the rest of the class, and send it to me. If you do all that, you fulfill your 10% course mark allocated to these sessions.

If you quickly succeed in solving a problem (some of them are very easy!), try one of the unsolved problems from previous sessions, which are given at the end.

New Problems

27. Define the following transformation on languages, similar to log(L) in the text, p. 76:

\[
\text{sqrt}(L) = \{ x \in \Sigma^* : \exists y \text{ such that } |y| = |x|^2 \text{ and } xy \in L \}.
\]

Show that if \( L \) is regular, then so is \( \text{sqrt}(L) \). Hint: modify the construction we used for \( \log(L) \).

28. Let \( M \) be an NFA. Show that the set of all strings in \( L(M) \) having exactly one accepting path is a regular language. Hint: instead of using Boolean matrix multiplication, use ordinary matrix multiplication to compute the number of accepting paths.

29. Two automata \( M_1, M_2 \) over the same input alphabet are said to be isomorphic if there is some permutation of the names of the states that changes \( M_1 \) into \( M_2 \). The Myhill-Nerode theorem implies that two minimal DFA’s accepting the same regular language are isomorphic. Show, by means of an example, this is not true for NFA’s.

30. What are the Myhill-Nerode equivalence classes of the language \( \{ a^m b^n : 1 \leq m \leq n \} \)?

31. Use the Myhill-Nerode theorem to show that the language

\[ \{ 0^m 1^n : \gcd(m, n) = 1 \} \]
is not regular. Hint: consider strings of the form $0^p$ where $p$ is a prime.

32. What are the Myhill-Nerode equivalence classes for the language \{a^{2^n} : n \geq 0\}? 

**Older Problems Unsolved from Previous Sessions**

2. Show that the number of bordered words of length $n$ is the same as the number of words of length $n$ that begin with a nonempty even-length palindrome. Hint: find a bijection. This one is hard!

3. Is the number of bordered words of length $n$ the same as the number of words of length $n$ that begin with a nonempty square? Prove or disprove. Hint: do some computation.

4. Show that $x$ is an even-length palindrome if and only if there exists a string $y$ such that $x = y \text{III} y^R$.

5. Call a language $L$ commutative if for all $x, y \in L$ we have $xy = yx$. Show that $L$ is commutative if and only if there exists a word $w$ such that $L \subseteq w^s$.

9. Prove or disprove: for all integers $m, n \geq 1$, $x^m$ is an antipalindrome if and only if $x^n$ is an antipalindrome.

12. When is the concatenation of two antipalindromes a palindrome? Give necessary and sufficient conditions. Possible strategy: do some experiments.

13. Find an expression for the perfect shuffle of two languages $L_1$ and $L_2$ in terms of operations like morphism, inverse morphism, and intersection. Hint: one approach is to modify the expression we found for ordinary shuffle. Conclude that if $L_1, L_2$ are regular then so is the perfect shuffle of $L_1$ and $L_2$.

14. Define $\text{min}(L) = \{ x \in L : \text{no proper prefix of } x \text{ is in } L \}$. Find an expression for $\text{min}(L)$ in terms of operations like quotient, complement, etc. Conclude that if $L$ is regular, so is $\text{min}(L)$.

16. Is it possible to avoid the pattern $xx'$, where $x'$ is a conjugate of $x$, over a 3-letter alphabet? (Remember that “avoid the pattern” means “does there exist an infinite word containing no occurrences of the pattern as a subword?”)

19. Is the following language regular? $\{xx^Rw : x, w \in \{0, 1\}^+\}$.

22. Show that if $M$ is an $n$-state DFA, and accepts at least one string that is a palindrome, then it accepts a palindrome of length $< 2n^2$. Hint: using $M$ design an NFA-$\epsilon$ to accept the first halves of the palindromes in $L(M)$.

24. Let $L$ be a regular language, and let $s$ be a substitution by regular languages. Must $s^{-1}(L) = \{ x : s(x) \subseteq L \}$ be regular?

25. Give an example of a unary nonregular language $L$, not containing $\epsilon$, for which $L^2$ is regular.