Here’s how these problem-solving sessions work.

Start by working on the problem indicated by the number on the front of this sheet. If you find a solution, let me know and sketch it for me. If it’s correct, you then present it on the board to the class. Next, you write up your solution within one week, in a format I can post for the rest of the class, and send it to me. If you do all that, you fulfill your 10% course mark allocated to these sessions.

If you quickly succeed in solving a problem (some of them are very easy!), try one of the unsolved problems from previous sessions, which are given at the end.

New Problems

48. How could you efficiently find the length of the shortest string generated by a given CFG?

49. If the shortest string generated by a CFG is of length $n$, how could you efficiently find the lexicographically least such string?

50. Let $w = 1010^210^410^8\cdots$. Let $L$ be the language of all prefixes of $w$. Is $\overline{L}$ context-free? Prove or disprove.

51. If $\text{shuff}(L, \{0\})$ is a CFL, need $L$ be a CFL? Here ‘shuff’ means the not-necessarily-perfect shuffle.

Older Problems Unsolved from Previous Sessions

2. Show that the number of bordered words of length $n$ is the same as the number of words of length $n$ that begin with a nonempty even-length palindrome. Hint: find a bijection. This one is hard!

3. Is the number of bordered words of length $n$ the same as the number of words of length $n$ that begin with a nonempty square? Prove or disprove. Hint: do some computation.

4. Show that $x$ is an even-length palindrome if and only if there exists a string $y$ such that $x = y \overline{\overline{y}} y^R$. 
5. Call a language $L$ **commutative** if for all $x, y \in L$ we have $xy = yx$. Show that $L$ is commutative if and only if there exists a word $w$ such that $L \subseteq w^*$.

9. Prove or disprove: for all integers $m, n \geq 1$, $x^m$ is an antipalindrome if and only if $x^n$ is an antipalindrome.

12. When is the concatenation of two antipalindromes a palindrome? Give necessary and sufficient conditions. Possible strategy: do some experiments.

13. Find an expression for the *perfect shuffle* of two languages $L_1$ and $L_2$ in terms of operations like morphism, inverse morphism, and intersection. Hint: one approach is to modify the expression we found for ordinary shuffle. Conclude that if $L_1, L_2$ are regular then so is the perfect shuffle of $L_1$ and $L_2$.

14. Define

$$\text{min}(L) = \{x \in L : \text{no proper prefix of } x \text{ is in } L\}.$$  

Find an expression for $\text{min}(L)$ in terms of operations like quotient, complement, etc. Conclude that if $L$ is regular, so is $\text{min}(L)$.

19. Is the following language regular? $\{xx^Rw : x, w \in \{0, 1\}^+\}$.

22. Show that if $M$ is an $n$-state DFA, and accepts at least one string that is a palindrome, then it accepts a palindrome of length $< 2n^2$. Hint: using $M$ design an NFA-$\epsilon$ to accept the first halves of the palindromes in $L(M)$.

24. Let $L$ be a regular language, and let $s$ be a substitution by regular languages. Must $s^{-1}(L) = \{x : s(x) \subseteq L\}$ be regular?

33. Show that if an $n$-state NFA accepts a string $w$ by at least two different acceptance paths, then it accepts such a string with $|w| < n^2 + n$. How tight is this bound?

35. What are the minimal elements for the primes (starting with $p_1 = 2$) represented in base 5? A computer will be helpful.

37. Give an efficient algorithm for the following problem: given a finite set of words $S$, is there a word $x$ such that $x$ can be expressed in two different ways as a concatenation of elements of $S$? Here by efficient we mean polynomial in the sum of the lengths of the words of $S$. Hint: use an NFA and a graph algorithm.

39. Show that if $L_2 = \{x \in \{0, 1\}^* : x \neq yy \text{ for all } y\}$, then $L_2$ is context-free.

40. Show that if $L_3 = \{(a^nb^n)^n : n \geq 1\}$, then $\overline{L_3}$ is context-free.

41. Show that if $L_4 = \{x \in \{0, 1\}^* : x \text{ is a finite prefix of } t\}$, where $t$ is the Thue-Morse word, then $\overline{L_4}$ is context-free.

43. Show that if $L$ is a CFL, then the language $\{x : xx^R \in L\}$ need not be a CFL.

47. Prove or disprove:

$$L_3 = \{a^ib^ja^ib^j : i, j \geq 1\}$$  

is the intersection of two CFL’s.