Today we proved Theorem 2.5.1 using a proof different from that in the text. Here it is:

We define $\overline{0} = 1$ and $\overline{1} = 0$.

**Lemma 1.** Let $\mu$ be the morphism defined by $\mu(0) = 01$ and $\mu(1) = 10$. Then $\mu(t_n) = t_{2n}t_{2n+1}$.

**Proof.** By induction on $n$.

The base case is $n = 0$. Then we have $\mu(t_0) = \mu(0) = 01 = t_0t_1$.

For the induction step, assume the result is true for $i < n$; we prove it for $i = n$.

Now the binary expansion of $2n$ is the same as that for $n$, except with an extra 0 on the end. The binary expansion of $2n + 1$ is that same as that for $n$, except with an extra 1 on the end. Hence $t_{2n} = t_n$ and $t_{2n+1} = \overline{t_n}$.

It follows that $\mu(t_n) = t_n \overline{t_n} = t_{2n}t_{2n+1}$. \(\square\)

Note that this lemma implies that

- If $t_r = a$ and $r$ is even, then $t_{r+1} = \overline{a}$;
- If $t_r = a$ and $r$ is odd, then $t_{r-1} = \overline{a}$.

**Theorem 2.** The Thue-Morse word $t$ is overlap-free.

**Proof.** We assume that $t$ has an overlap $axaxa$ beginning at position $k$, with $|ax| = n$. Thus it looks like

$$t = t_0t_1 \cdots t_{k-1} \overline{a} \overline{t_k} t_{k+1} \cdots t_{k+n-1} \overline{a} t_{k+n} \overline{t_{k+n+1}} \cdots t_{k+2n-1} \overline{a} t_{k+2n} \cdots .$$

To get a contradiction we assume that (a) this overlap is smallest among all overlaps in $t$ and (b) among all overlaps of this size, it appears earliest in $t$. In other words, we assume that $n$ is as small as possible and $k$ as small as possible for this $n$.

There are a number of cases to consider:

Case 1: $k$ even, $n$ even.

Since $k + 2n$ is even, we get $t_{k+2n+1} = \overline{t_{k+2n}} = \overline{a}$.  

Letting $u = t_{\lfloor k / 2 \rfloor + 1..\lfloor k / 2 \rfloor + n - 1}$ and $v = t_{\lfloor k / 2 \rfloor + n/2 + k/2 + n - 1}$, by the lemma we see $\mu(auava) = axaxa = t[k..k + 2n + 1].$ Furthermore, since $\mu(au) = ax = \mu(av)$, we must have $u = v$. So $auua = t_{\lfloor k / 2 \rfloor + n}^k$ is an overlap with $|au| = n/2 < n$, a contradiction.

Case 2: $k$ odd, $n$ even.

Since $k$ is odd, we have $t_{k-1} = \overline{\alpha}$. Similarly, since $k + n$ and $k + 2n$ are both odd, we get $t_{k+n-1} = \overline{\alpha}$ and $t_{k+2n-1} = \overline{\alpha}$. So Eq. (1) above can be rewritten as

$$t = t_0t_1 \cdots t_{k-1} \overline{\alpha} \overline{t_k} \overline{t_{k+1}} \overline{t_{k+n-2}} \overline{t_{k+n}} \overline{t_{k+n+1}} \overline{t_{k+2n-2}} \overline{t_{k+2n}} \overline{t_{k+2n+1}} \overline{t_{k+2n+2}} \cdots, \quad (2)$$

where $x = y\overline{\alpha}$.

Now $t[k - 1..k + 2n - 1] = \overline{\alpha}y\overline{\alpha}y\alpha\overline{\alpha}$ is an overlap of the same length as before, but occurring one place earlier than before, that is, $k - 1 < k$. This is a contradiction.

Case 3: $k$ even, $n$ odd.

There are two subcases here: $n = 1$ and $n > 1$. If $n = 1$ then the overlap is just $aaa$. But from the lemma we know that $t \in (01 + 10)^\omega$, so we cannot have three consecutive identical symbols in $t$.

The second subcase is $n > 1$. The idea here is to “ping-pong” back and forth between the two copies of $x$, learning more and more symbols of $x$.

We start with the first copy.

Since $k + n - 1$ is even, the lemma gives us $t_{k+n-1} = \overline{t_{k+n}} = \overline{\alpha}$. So the last symbol of $x$ must be $\overline{\alpha}$, which tells us that in the second copy of $x$ we have $t_{k+2n-1} = \overline{\alpha}$. Now $k + 2n - 1$ is odd, so the lemma gives us $t_{k+2n-2} = \overline{t_{k+2n-1}} = a$. Back in the first copy of $x$, this gives us $t_{k+n-2} = a$.

Now $k + n - 2$ is odd, so the lemma gives us $t_{k+n-3} = \overline{t_{k+n-2}} = \overline{\alpha}$. In the second copy of $x$, we get $t_{k+2n-3} = \overline{\alpha}$, too.

We continue ping-ponging back and forth, learning more and more symbols of $x$. You can see that the situation above continues, giving

$$x = \cdots a\overline{\alpha}a\overline{\alpha}a\overline{\alpha}a\overline{\alpha}a\overline{\alpha} \cdots.$$

But since $n = |ax|$ is odd, this means that $ax$ must begin and end with the same symbol. But $ax$ begins with $a$ and ends with $\overline{\alpha}$, a contradiction.

Case 4: $k$ odd, $n$ odd.

This is just like the previous case. In the second subcase, we “ping-pong” back and forth in the same manner, except that we start with the second copy.

Thus, all cases give us a contradiction, so there is no overlap in $t$. \qed