1. [10 marks] There is a slightly different version of Kolmogorov complexity $K(x)$, which is defined as the length of the shortest program-input pair $(P, i)$ such that running $P$ on $i$ gives $x$, but demanding that the encoding of the pair $(P, i)$ be prefix-free: no encoding is a prefix of any other. The point of this is that if we concatenate two such encodings $(P, i)(Q, j)$ we can always tell where the first ends and the second begins.

For this version of Kolmogorov complexity prove that $K(xy) \leq K(x) + K(y) + O(1)$, where the constant in the big-$O$ does not depend on $x$ or $y$.

2. [10 marks] Let $r$ be an arbitrary regular expression and $G$ be an arbitrary grammar. Let $L(r)$ (resp., $L(G)$) be the corresponding languages. Which of the following two problems is solvable and which is unsolvable? Justify.

   (a) Decide if $L(r) \subseteq L(G)$ ;
   (b) Decide if $L(G) \subseteq L(r)$.

3. [10 marks] The point of this exercise is to show that concatenation can dramatically decrease Kolmogorov complexity.

   Let $C(x)$ denote Kolmogorov complexity. Give an example of a family of binary strings $x_n$ and $y_n$ such that $C(x_n y_n)/\min(C(x_n), C(y_n))$ tends to 0 as $n \to \infty$. Justify.