All answers should be accompanied by proofs. In all problems the underlying alphabet $\Sigma$ is assumed to be finite.

1. [10 marks] Fix an alphabet $\Sigma$. Let $x, z \in \Sigma^*$ be strings. Show that no matter what $x, z$ are, the language $L_{x,z} := \{ y \in \Sigma^* : xy = yz \}$ is a regular language.

2. [10 marks] Let $w$ be a nonempty word of length $n$. Show that $w$ is primitive if and only if $w$ has $n$ distinct conjugates.

3. [10 marks] Call a word $w$ odd if every nonempty subword has the property that at least one letter appears an odd number of times. For example, $abac$ is odd, but $cabcacbc$ is not.

   (a) [5 marks] Show that if $w$ is an odd word over an alphabet with $k$ letters, then $|w| < 2^k$. (Hint: it is possible, but surprisingly difficult, to prove this by induction on $k$. I suggest trying to find a proof not using induction.)

   (b) [5 marks] Prove that the bound in (a) is sharp, by constructing, for each integer $k \geq 1$, an odd word of length $2^k - 1$ over an alphabet of size $k$. (This direction is easier, and induction is one good way to prove it.)