# University of Waterloo
# CS 462 — Formal Languages and Parsing
# Winter 2018
# Problem Set 3

*Distributed Monday, January 22 2018.*

*Due Monday, January 29 2018 by 5 PM. Hand in to LEARN.*

*All answers should be accompanied by proofs.* In all problems the underlying alphabet $\Sigma$ is assumed to be finite.

1. [10 marks] Let $(n)_{10}$ be the string over the alphabet $\{\mathtt{0}, \mathtt{1}, \mathtt{2}, \mathtt{3}, \mathtt{4}, \mathtt{5}, \mathtt{6}, \mathtt{7}, \mathtt{8}, \mathtt{9}\}$ giving the decimal representation of the integer $n$. Define

$$L = \{(n)_{10} \ : \ \pi \text{ has } n \text{ consecutive 1's somewhere in its decimal expansion } \}.$$

For example, because $\pi = 3.14$, certainly $\mathtt{1} \in L$. The website http://www.angio.net/pi/ lets you search for patterns in $\pi$.

Prove that $L$ is regular. Hint: this is more or less a "trick" question, and can be answered without much knowledge about $\pi$.

2. [10 marks] Define a map on words $w \in \Sigma^*$ called the *shift* $S(w)$, as follows:

$$S(w) := \begin{cases} w, & \text{if } w = \epsilon; \\ ax, & \text{if } w = xa \text{ for } a \in \Sigma. \end{cases}$$

Basically $S$ just shifts the symbols in $w$ to the right one position, moving the symbol at the end of $w$ to the front. So, for example, $S(\mathtt{eighth}) = \mathtt{height}$.

Define $S^n(w)$ for $n \geq 0$ by

$$S^n(w) := \overbrace{S(S(\cdots S(}^{n} w) \cdots)).$$

(Or more formally, $S^0(w) = w$ and $S^n(w) = S(S^{n-1}(w))$ for $n \geq 1$.) Thus, $S^n(w)$ basically shifts $w$ to the right $n$ positions, moving the last $n$ symbols of $w$ to the front. For example, $S^2(\mathtt{centre}) = \mathtt{recent}$. Notice that this is defined even if $n > |w|$. For example, $S^4(\mathtt{at}) = \mathtt{at}$.

Extend $S$ and $S^n$ to languages $L$ as follows:

$$S(L) = \bigcup_{w \in L} S(w)$$
$$S^n(L) = \bigcup_{w \in L} S^n(w).$$

(a) Show that if $L$ is regular, then so is $S(L)$.

(b) Give an example of a regular language $L$ such that the languages $S^n(L)$, for $n \geq 0$, are all distinct.

3. [10 marks] In the text, Theorem 3.4.3, we prove that if $L$ is regular, then so is the language

$$\text{cyc}(L) := \{x_1 x_2 \ : \ x_2 x_1 \in L\}.$$

Prove this in a completely different way, based on the following idea: let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA accepting $L$, and write $\text{cyc}(L)$ somehow in terms of the language accepted by $M_{pq}$, where

$$M_{pq} := (Q, \Sigma, \delta, p, \{q\})$$

for $p, q \in Q$.