1. [10 marks] Let \((n)_{10}\) be the string over the alphabet \(\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}\) giving the decimal representation of the integer \(n\). Define

\[ L = \{(n)_{10} : \pi \text{ has } n \text{ consecutive 1’s somewhere in its decimal expansion } \}. \]

For example, because \(\pi = 3.14\), certainly \(1 \in L\). The website \(\text{http://www.angio.net/pi/}\) lets you search for patterns in \(\pi\).

Prove that \(L\) is regular. Hint: this is more or less a “trick” question, and can be answered without much knowledge about \(\pi\).

2. [10 marks] Define a map on words \(w \in \Sigma^*\) called the \textit{shift} \(S(w)\), as follows:

\[
S(w) := \begin{cases} 
  w, & \text{if } w = \epsilon; \\
  ax, & \text{if } w = xa \text{ for } a \in \Sigma.
\end{cases}
\]

Basically \(S\) just shifts the symbols in \(w\) to the right one position, moving the symbol at the end of \(w\) to the front. So, for example, \(S(\text{eighth}) = \text{height}\).

Define \(S^n(w)\) for \(n \geq 0\) by

\[
S^n(w) := \underbrace{S(S(\cdots S}(w)\cdots)).
\]

(Or more formally, \(S^0(w) = w\) and \(S^n(w) = S(S^{n-1}(w))\) for \(n \geq 1\).) Thus, \(S^n(w)\) basically shifts \(w\) to the right \(n\) positions, moving the last \(n\) symbols of \(w\) to the front. For example, \(S^2(\text{centre}) = \text{recent}\). Notice that this is defined even if \(n > |w|\). For example, \(S^4(\text{at}) = \text{at}\).
Extend $S$ and $S^n$ to languages $L$ as follows:

\[
S(L) = \bigcup_{w \in L} S(w)
\]

\[
S^n(L) = \bigcup_{w \in L} S^n(w).
\]

(a) Show that if $L$ is regular, then so is $S(L)$.

(b) Give an example of a regular language $L$ such that the languages $S^n(L)$, for $n \geq 0$, are all distinct.

3. [10 marks] In the text, Theorem 3.4.3, we prove that if $L$ is regular, then so is the language

\[
cyc(L) := \{x_1x_2 : x_2x_1 \in L\}.
\]

Prove this in a completely different way, based on the following idea: let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA accepting $L$, and write $cyc(L)$ somehow in terms of the language accepted by $M_{pq}$, where

\[
M_{pq} := (Q, \Sigma, \delta, p, \{q\})
\]

for $p, q \in Q$. 

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