All answers should be accompanied by proofs. In all problems the underlying alphabet \( \Sigma \) is assumed to be finite.

1. [10 marks] Let \( h : \Sigma^* \to \Sigma^* \) be a morphism, and let \( L \subseteq \Sigma^* \) be a language. Define

\[
h^{-*}(L) = \bigcup_{i \geq 0} h^{-i}(L),
\]

where by \( h^{-i}(L) \) we mean \( h^{-1}(h^{-1}(\cdots h^{-1}(L) \cdots)) \). Note that \( h^{-0}(L) = L \). Prove that if \( L \) is regular, so is \( h^{-*}(L) \).

2. [10 marks] Let \( L \) be a language, and define

\[
bord(L) = \{ u \in \Sigma^+ : \exists x \in \Sigma^* \text{ such that } uxu \in L \}.
\]

(a) Show, by explicitly constructing a finite automaton, that if \( L \) is regular then so is \( bord(L) \).

(b) Use your proof in (a) to show that there exists a function \( f(n) \) such that if an \( n \)-state DFA \( M \) accepts at least one bordered word, then \( M \) accepts some bordered word of length at most \( f(n) \). Be as explicit as possible in defining your \( f \).

3. [10 marks] For each \( n \geq 1 \), consider the language \( L_n \) defined over the alphabet \( \Sigma_n \cup \{\#\} \), where \( \Sigma_n = \{1, 2, 3, \ldots, n\} \), as follows:

\[
L_n = \{ \#w\#k\#w[k]\# : |w| \geq n \text{ and } w \in \Sigma_n^* \text{ and } 1 \leq k \leq n \}.
\]

Thus, for example, the language \( L_3 \) contains the strings \#3123\#2\#1\# and \#1111\#2\#1, but not the string \#123\#3\#1\#.

Show that \( L_n \) can be accepted by a 2DFA of \( O(n) \) states. Explain your 2DFA in words, justify it in words, and draw it for \( n = 4 \), but a complete formal proof is not necessary.