All answers should be accompanied by proofs. In all problems the underlying alphabet $\Sigma$ is assumed to be finite.

1. [10 marks] Let $L \subseteq \Sigma^*$ be an arbitrary language, not necessarily regular. Let $\overline{L} = \Sigma^* - L$ be the complement of $L$. Consider the two Myhill-Nerode equivalence relations $R_L$ and $R_{\overline{L}}$. What is the relationship between the equivalence classes of these two relations? Be as specific as possible.

2. [10 marks] Define $\text{PAL}$ to be the language of palindromes over $\{0, 1\}$. Show that for the Myhill-Nerode equivalence relation $R_{\text{PAL}}$, every string in $\{0, 1\}^*$ is in its own equivalence class.

3. [10 marks] A string of length $n$ over the alphabet $\{0, 1, 2, \ldots, n - 1\}$ is called Bell if it has the property that the first letter is 0, and each succeeding letter is at most 1 more than the maximum of all preceding letters. For example, 0010 2 is Bell, but 00201 is not (since the symbol 2 violates the condition).

(By the way, the number of such length-$n$ strings is counted by the Bell numbers — see the Wikipedia article http://en.wikipedia.org/wiki/Bell_number — hence the name.)

Let $B_n$ be the (finite) language of all Bell strings of length $n$.

Here is $B_3$: \{000, 001, 010, 011, 012\}.

(a) [5 marks] Show that $B_n$ can be accepted by a DFA with $O(n^2)$ states.

(b) [5 marks] Using the Myhill-Nerode theorem, show that every DFA accepting $B_n$ has $\Omega(n^2)$ states.

4. [10 marks — extra credit only] Find good nontrivial upper and lower bounds on the size of the shortest regular expression for the language $B_n$ of problem 3 above. I know there is an expression of size $O(2^n)$, but can you do significantly better? (We get to decide what constitutes a “good nontrivial” solution.)