All answers should be accompanied by proofs. In all problems the underlying alphabet $\Sigma$ is assumed to be finite.

1. [10 marks] A state $q$ of a DFA $M = (Q, \Sigma, \delta, q_0, F)$ is sometimes called “dead” if $\delta(q, x) \not\in F$ for all $x \in \Sigma^*$.

   (a) [5 marks] Prove that a minimal DFA has at most one dead state.
   (b) [5 marks] Give an example of a minimal DFA with input alphabet $\Sigma = \{0, 1\}$, but without any dead states.

2. [10 marks] Using any method discussed in class, find the minimal DFA equivalent to the one whose transition diagram is given below. Show all steps.
3. [10 marks] A string of length $n$ over the alphabet $\{0, 1, 2, \ldots, n-1\}$ is called Bell if it has the property that the first letter is 0, and each succeeding letter is at most 1 more than the maximum of all preceding letters. For example, 00102 is Bell, but 00201 is not (since the symbol 2 violates the condition).

(By the way, the number of such length-$n$ strings is counted by the Bell numbers — see the Wikipedia article http://en.wikipedia.org/wiki/Bell_number — hence the name.)

Let $B_n$ be the language of all Bell strings of length $n$.

Here is $B_3$: \{000, 001, 010, 011, 012\}.

(a) [5 marks] Show that $B_n$ can be accepted by a DFA with $O(n^2)$ states.

(b) [5 marks] Using the Myhill-Nerode theorem, show that any DFA accepting $B_n$ has at least $\Omega(n^2)$ states.

4. [10 marks — extra credit only] Find good nontrivial upper and lower bounds on the size of the shortest regular expression for the language $B_n$ of problem 3 above. I know there is an expression of size $O(2^n)$, but can you do significantly better? (I get to decide what constitutes a “good nontrivial” solution.)