1. [10 marks] Suppose you are given a set of horizontal line segments, where segment \( s \) goes from point \((r_s, y_s)\) to point \((l_s, y_s)\). Assume that the segments have distinct \(x\)-coordinates and distinct \(y\)-coordinates. The set of segments determines a partition of the plane into rectangles formed by extending a vertical line segment upward and downward from each endpoint of each horizontal segment until it hits some other horizontal segment. See the Figure below.

Suppose you were to build this structure by adding the horizontal segments one by one. One issue will be: How many vertical segments are cut by the new horizontal segment? This depends on the ordering of the segments. In the Figure below, if we add the long segment last, it cuts 5 vertical segments of the previous structure (causing them to be truncated), whereas if we add the top segment last, it cuts 0 vertical segments of the previous structure.

(a) Give an example to show an ordering of \( n \) segments such that \( \Theta(n) \) of the segments each cut \( \Theta(n) \) rectangles.

(b) Use backwards analysis to prove that if you add the segments in random order then the expected number of vertical segments cut by a new horizontal segment is constant.

(You do not need to give the whole build algorithm and you do not need to worry about other issues, like how to figure out where the new segment lies with respect to the others.)

2. [15 marks] A graph \( G = (V, E) \) is 3-colourable if there is a partition of \( V \) into 3 sets \( A_0, B_0, C_0 \) such that no edge has both endpoints in the same set. We say that a \( G \) has a bipartition without monochromatic triangles if we can partition \( V \) into two sets \( A \) and \( B \) so that no triangle has all three vertices in the same set.
Observe that if $G$ is 3-colourable, then it has a bipartition without monochromatic triangles: if the 3 colour classes are $A_0, B_0, C_0$ then we can reassign the vertices in $C_0$ into $A_0$ and $B_0$ (arbitrarily) to get the desired bipartition.

This problem is about a randomized algorithm to find a partition without monochromatic triangles in a 3-colourable graph. (We cannot do this by first finding a 3-colouring, because that’s NP-hard.)

The algorithm starts from an arbitrary bipartition of $V$ into two sets $A, B$. Then the algorithm repeats the following step some number of times: If there is a monochromatic triangle, then randomly pick one of its vertices and change its set.

In this problem you will fill in the details of this algorithm, under the assumption that the input graph $G$ is 3-colourable with the colour classes $A_0, B_0, C_0$. The analysis follows the same pattern as the analysis of the randomized 2-SAT algorithm given in class.

Define the score of any partition $A, B$ to be $s(A, B) = |A \cap A_0| + |B \cap B_0|$. Let $t = |A_0| + |B_0|$.

(a) Prove that if $s(A, B) = t$ then there are no monochromatic triangles.

(b) Prove that if $s(A, B) = 0$ then there are no monochromatic triangles.

(c) Suppose we take one step of the algorithm. How does $s(A, B)$ change, and what are the probabilities of each possibility?

(d) Model this as a Markov chain.

(e) Assume that there is a bound of $O(t^2)$ on the expected number of steps to reach $s(A, B) = 0$ or $n$. (This is true and interesting to prove, but you don’t have to do it for this assignment.) Finish giving the details of the algorithm to guarantee expected polynomial run time and probability at most $1/2$ that we don’t find the desired bipartition.