CS 466/666 Fall 2018 due Tuesday November 13, 5 PM

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ASSIGNMENT 6

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1. [10 marks] Recall that in a graph $G = (V, E)$ a set $U \subseteq V$ is an independent set iff $V - U$ is a vertex cover. Also, U is an independent set in G iff U is a clique in the complement of G. These relationships provide polynomial time reductions among these three NP-complete problems. The point of this question is to ask whether these reductions preserve good approximations.

   (a) [5 marks] Does a polynomial time constant factor approximation algorithm for the Independent Set Problem give a polynomial time constant factor approximation algorithm for the Clique Problem? Explain.

   (b) [5 marks] Does a polynomial time constant factor approximation algorithm for the Independent Set Problem give a polynomial time constant factor approximation algorithm for the Vertex Cover Problem? Explain.

2. [10 marks] In class, we studied an approximation algorithm for weighted vertex cover using linear programming and rounding. This problem is about an alternate algorithm. Recall that for the weighted vertex cover, we are given a graph $G = (V, E)$ and a weight $w(v) \geq 0$ for each vertex $v \in V$. The goal is to find a vertex cover $U \subseteq V$ that minimizes $w(U) = \sum_{u \in U} w(u)$.

   Here is the algorithm:

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while there is an edge e=(u,v) such that neither u nor v has weight 0
    let x = min{w(u), w(v)}
    w(u) = w(u) - x
    w(v) = w(v) - x
return the set of vertices of weight 0
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   The goal of this problem is to prove that the algorithm is a 2-approximation.

   (a) [3 marks] What is the running time of the algorithm? You may wish to fill in some implementation details to obtain a fast run time.

   (b) [1 mark] In the unweighted case (i.e., all vertices have weight 1), we proved correctness by arguing that the set of edges chosen by the algorithm form a matching. Show (by an example) that this is no longer true for general weights.

   Let A be the vertex cover found by the algorithm and let OPT be an optimum vertex cover. Let w_0 be the initial weight function on the vertices, and let w_i be the weight function on the vertices after i executions of the main loop. Suppose that in the i^{th} execution of the loop we choose edge $e_i = (u_i, v_i)$, and x has the value x_i. Suppose the algorithm finishes with $w_t(A) = 0$. Given this notation, the goal is to prove $w_0(A) \leq 2w_0(\text{OPT})$.

   (c) [6 marks] Prove by induction on $t - i$ (with base case $t - i = 0$) that $w_i(A) \leq 2w_i(\text{OPT})$. Hint: relate $w_i(A)$ to $w_{i+1}(A)$ and relate $w_i(\text{OPT})$ to $w_{i+1}(\text{OPT})$.