1. [10 marks] Given a graph $G = (V, E)$ a proper colouring assigns colours to the vertices such that for any edge, its endpoints have different colours.

   (a) [5 marks] Give a greedy algorithm to find a proper colouring of $G$ with $\Delta + 1$ colours, where $\Delta$ is the maximum vertex degree in $G$.

   (b) [5 marks] Give an algorithm to find a proper colouring a 3-colourable graph with $O(\sqrt{n})$ colours. **Hint:** Combine the following two ideas. If $\Delta \leq \sqrt{n}$, use part (a). If some vertex $v$ has degree $> \sqrt{n}$, show how to use 3 colours to properly colour $v \cup N(v)$. Here $N(v)$ is the *neighbourhood* of $v$ that consists of all vertices adjacent to $v$.
   
   **Fact:** It is NP-hard to find a proper 4-colouring of a 3-colourable graph.

2. [10 marks] Consider the following optimization problem. Given a set $X$ of positive integers, split $X$ into disjoint subsets $A$ and $B$ such that $A \cup B = X$ and $\max\{\sum A, \sum B\}$ is as small as possible. This problem is NP-hard. Consider the following polynomial-time approximation algorithm: Sort the elements of $X$ in increasing order $x_1 \leq x_2 \leq \cdots \leq x_n$. For $i = 1, \ldots, n$ put $x_i$ into whichever of $A$ or $B$ currently has smaller sum, breaking ties arbitrarily.

   (a) [8 marks] Prove that this approximation algorithm has approximation factor 1.5.

   (b) [2 marks] Give an example to show that this approximation factor is tight.