PTAS - polynomial time approximation scheme. Recall - vertex cover, more general: set cover.

Today - packing.

Given elements 1...n

Sets $S_1 \ldots S_k$

$S_i \subseteq \{1, \ldots, n\}$

find max. number of $S_i$'s s.t. no two intersect.

Graph version:

Independent Set problem - Given graph $G = (V, E)$

find max size subset $U \subseteq V$ s.t. no edge $e = (u, v)$ has both endpoints in $U$.

\[ \times \text{ Ind. Set is special case of Set Packing} \]

where edges incident to $v_i$

= Set $S_i$

Ind. Set is NP-complete. \[ \therefore \text{ Set Packing is NP-complete.} \]

FACT: No $n^{1-\epsilon}$ factor approx. for Ind. Set (unless $P=NP$)

Special geometric case of Set Packing. (still NP-hard)

Given $n$ unit squares in $\mathbb{R}^2$ find max. number s.t.

no two intersect

\[ \begin{array}{ccc}
1 & 2 & 3 \\
\hline
4 & 5 & 6 \\
\end{array} \]

can get 3 pairwise disjoint

but not 4.

As ind. set:

\[ \begin{array}{ccc}
1 & 2 & 3 \\
\hline
4 & 5 & 6 \\
\end{array} \]
Applications:
- VLSI
-map labelling

$k$ cities
$4k$ possible labels ($4$ per city)
pick $k$ labels that do not intersect.

Simple constant factor approx. for packing unit squares
- pick a square, throw away those that intersect
Repeat.

\[
\text{opt} = 4 \quad \text{alg} = 1
\]

\[A \geq \alpha \cdot \text{opt} \quad \text{for } \alpha \text{-approx.}\]

Claim This is a $\frac{1}{4}$-approx alg.

Pf. Consider OPT each square in OPT intersects some square in \(A\)

Each square of alg. intersects \(\leq 4\) squares of OPT

\[\text{OPT} \leq 4 \cdot A\]

Equivalent formulation:
Square \(\rightarrow\) center point

Pick \(\max \#\) pts s.t.
for any two, \(P, Q\):
\[|p.x - q.x| > 1\]
or \[|p.y - q.y| > 1\]
Grid Approx Alg. \[ \text{Input: set } P \text{ of } n \text{ points} \]

- put down a unit grid. Note $n \times n$ grid since a gap $\geq 1$ in $x$-coords gives separate subproblems
- Let $R_0 = \text{shaded squares}$
  \[ = \{ (x, y) : \text{lx, ly even} \} \]
  opt. soln for $R_0 \cap P$
- take one point from each shaded square (if there is one)

- \[ R_1 = \{ (x, y) : \text{lx odd, ly even} \} \]
- \[ R_2 = \{ (x, y) : \text{lx even, ly odd} \} \]
- \[ R_3 = \{ (x, y) : \text{lx odd, ly odd} \} \]

Alg. For $i = 0, 1, 2, 3$, let $Q_i = \text{opt soln in } R_i \cap P$

- let $Q = \text{largest } Q_i$
- $|Q| \geq \frac{1}{4} \sum_{i=0}^{3} |Q_i|$ max. $\geq \text{avg.}$

Claim. This is a $\frac{1}{4}$-approx. alg. \[ \text{If: } \text{OPT} = \frac{3}{n} \log \text{OPT } \cap \bigcap_{i=0}^{3} R_i \leq \frac{3}{n} \sum_{i=0}^{3} |Q_i| \leq 4 |Q| \]

\[ \therefore |Q| \geq \frac{1}{4} \text{OPT} \]

We can do better than $\frac{1}{4}$ — in fact arbitrarily close to 1

Shift grid approach. [Hochbaum & Maas, '85]

- fix $k \geq 2$ (k = 2 is above)
- let $R_{ij} = \{ (x, y) : \text{lx mod } k \neq i, \text{ly mod } k \neq j \}$
  \[ 0 \leq i, j \leq k-1 \]

Ex. $k = 3$

*** How many points? 4 at most. ***
There are \( k^2 \) \( R_{ij} \)'s.
Each point is in \( (k-1)^2 \) of them (we can exclude any of \( k-1 \) rows, any of \( (k-1) \) columns).

\[
\text{# of shaded squares for one } R_{ij} \leq \frac{n^2}{k^2} \cdot \frac{n}{k} = \frac{n^2}{k^2}
\]

\[
\text{# points we can choose in one shaded square } \leq (k-1)^2
\]

Lemma: We can solve problem optimally in \( R_{ij} \cap P \) in poly. time (for constant \( k \)).

Pf. shaded squares are independent.

At most \( (k-1)^2 \) pts we can choose here.

Try all possible subsets of \( (k-1)^2 \) points. Check a subset in \( O(k^4) \)

\[
O(n^{(k-1)^2}) \text{ subsets in each shaded square}
\]

Time \( O(\text{poly}(n,k) \cdot n^{(k-1)^2}) \) for all shaded squares.

Algorithm

for \( i = 0 \ldots k-1, \ j = 0 \ldots k-1 \)

\( Q_{ij} = \text{opt. soln in } R_{ij} \cap P \)

\( Q = \max_{ij} Q_{ij} \)

Run Time: \( O(\text{poly}(n,k) \cdot n^{(k-1)^2}) \)

Analysis of approx.

\( \star \quad \|Q\| \geq \frac{1}{k^2} \sum_{ij=0}^{k-1} |Q_{ij}| \quad (\max \geq \text{avg}) \)

\[
(k-1)^2 \text{OPT} = \sum_{i,j} \text{opt} \cap R_{ij} \leq \sum_{ij} |Q_{ij}| \leq k^2 \|Q\|
\]

\( \uparrow \text{each point in } (k-1)^2 \text{ } R_{ij} \)'s \( \Rightarrow \)

So \( \|Q\| \geq \frac{(k-1)^2}{k^2} \text{OPT} \).
Approx. factor is \( \frac{(k-1)^2}{k^2} \)
\[ \begin{array}{cccccc}
   k & 2 & 3 & 5 & 10 & 100 \\
   \text{approx factor} & \frac{1}{4} & \frac{4}{9} & 0.64 & 0.81 & 0.9 \\
   \rightarrow \text{arbitrarily close to 1.} \\
\end{array} \]

**Definitions:**

**Approximation Scheme** - alg. \( A \), input \( I \) and parameter \( \varepsilon \)
- for \( \text{min. problem} \) \( A(I, \varepsilon) \leq (1 + \varepsilon) \cdot \text{OPT}(I) \)
- for \( \text{max. problem} \) \( A(I, \varepsilon) \geq (1 - \varepsilon) \cdot \text{OPT}(I) \)

**Poly. Time Approx. Scheme (PTAS)**
- for each fixed \( \varepsilon \), \( A \) runs in poly. time in \( n = |I| \)
  - e.g. \( O(n^{\frac{1}{\varepsilon}}) \)

**Fully Poly. Time Approx. Scheme (FPTAS)**
- \( A \) runs in time \( \text{poly. in } n = |I| \) and \( \frac{1}{\varepsilon} \)
  - e.g. \( O(\frac{1}{\varepsilon}^2 n^3) \) but \( O(n^{\frac{1}{\varepsilon}}) \) not FPTAS.

For above example
- approx. factor = \( \frac{(k-1)^2}{k^2} = \frac{k^2 - 2k + 1}{k^2} = 1 - \frac{2k-1}{k^2} \)
  - So \( \varepsilon = \frac{2k-1}{k^2} \)
  - \( \varepsilon = \Theta\left(\frac{1}{k}\right) \)
  - \( k = \Theta\left(\frac{1}{\varepsilon}\right) \)

run time \( O(\text{poly}(n, k)) \cdot \frac{(k-1)^2}{k^2} n \cdot O\left(\frac{1}{\varepsilon^2}\right) \)

This is a PTAS but not an FPTAS.

Next week: example of FPTAS.