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Recall

- approx. scheme $\forall \varepsilon > 0$

  $A(I,\varepsilon) \geq (1-\varepsilon)\, OPT(I)$ for max. problem.

- PTAS – run time poly. in n for fixed ε
- FPTAS – "fully poly. time approx scheme"
  - run time is poly. in n and $\frac{1}{\varepsilon}$

We've seen PTAS – packing unit squares
– bin packing.

Today: FPTAS

Knapsack Problem Given objects $1 \cdots n$ each with size $s_i \in \mathbb{N}$ and profit $p_i \in \mathbb{N}$, given knapsack capacity B find $K \subseteq \{1, \cdots, n\}$, $\sum_{i \in K} s_i \leq B$ and maximize $\sum_{i \in K} p_i$

[decision version] is NP-complete.

Ex. Show $\left\{\begin{array}{l}\text{Subset Sum} \\ \text{or Partition}\end{array}\right.$ reduces to Knapsack.

A pseudo-polynomial time alg. for Knapsack.

Dynamic Programming.

idea: for each profit sum, find min. size.

subproblems:

Find $S(i,p)$ = min. size for subset of items $\{1 \cdots i\}$ of profit exactly p.

$i = 1 \cdots n \quad p = 1 \cdots \underbrace{n \cdot \max_i \{p_i\}}_{P}$

P – upper bound on total profit.

Alg.

initialize $\forall p = 1 \cdots P \quad S(1,p) = \begin{cases} s_1 & \text{if } p = p_1 \\ \infty & \text{otherwise} \end{cases}$

for $i = 2 \cdots n$

  for $p = 1 \cdots P$

    $S(i,p) = \min\{\underset{i \text{ not in set}}{S(i-1,p)}, \underset{i \text{ in set}}{S(i-1, p-p_i) + s_i}\}$

return max p such that $S(n,p) \le B$ = max profit of subset of size B.

run time.

$O(nP) \quad = O(n^2 \cdot \max_i \{p_i\})$

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pseudo-poly. time – run time depends on p_i's
not on size of p_i's which is $\log(p_i)$ = # bits.

Some NP-complete problems don't have pseudo-poly. time algs (unless P = NP).

e.g. TSP – still NP-complete with 0-1 weights (= Ham. cycle).

Two notions of "easy" NP-complete problem

1. a pseudo-poly. time algorithm
2. an FPTAS

An FPTAS for knapsack (use above pseudo-poly. time alg.)

Idea: If p_i's are small (few bits)
then pseudo-poly. time alg. is poly. time $O(n^2 \max_i \{p_i\})$

So, round p_i's to have few bits.

<u>Alg.</u> Given ε (want approx $\geq (1-\varepsilon)$ OPT.)
Given $s_1 \cdots s_n$, $p_1 \cdots p_n$, B.

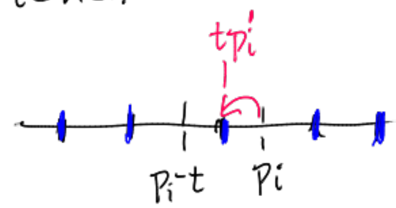
$\forall i \quad p_i' \leftarrow \lfloor \frac{p_i}{t} \rfloor$ $\quad t$ to be chosen.

Run dyn prog. alg. on p_i' — run time $O(n^2 \max_i \{p_i'\})$

Suppose result gives set $K(t) \subseteq \{1 .. n\}$
$K(t)$ is feasible i.e. $\sum_{i \in K(t)} s_i \leq B$.

We need to analyze $P(K(t)) = \sum_{i \in K(t)} p_i$
compared to $P(K^*) = \text{OPT}$

$$p_i - t \underset{(**)}{<} t p_i' \underset{(*)}{\leq} p_i$$



$$\sum_{i \in K(t)} p_i \underset{(*)}{\geq} \sum_{i \in K(t)} t p_i' \geq \sum_{i \in K^*} t p_i' \underset{(**)}{\geq} \sum_{i \in K^*} (p_i - t) = \underbrace{\sum_{i \in K^*} p_i}_{\text{OPT}} - t|K^*|$$

because $K(t)$ is opt for p_i'

$$\geq \text{OPT} - t|K^*| \geq \text{OPT} - t \cdot n = \text{OPT}\left(1 - \frac{t \cdot n}{\text{OPT}}\right)$$

note: $\text{OPT} \geq \max\{p_i\}$ $\qquad \geq \text{OPT}\left(1 - \frac{t \cdot n}{\max\{p_i\}}\right)$

want $\geq \text{OPT}(1-\varepsilon)$

so choose t s.t. $\varepsilon = \frac{t \cdot n}{\max\{p_i\}}$ i.e. $t = \frac{\varepsilon \cdot \max p_i}{n}$

Run time $O(n^2 \max p_i')$

$$p_i' = \left\lfloor \frac{p_i}{t} \right\rfloor = \left\lfloor \frac{n\, p_i}{\varepsilon \max p_i} \right\rfloor \le \frac{n}{\varepsilon}$$

So run time $O(n^3 \cdot \frac{1}{\varepsilon})$ — FPTAS

Best known alg. has run time $O(n \log \frac{1}{\varepsilon} + (\frac{1}{\varepsilon})^4)$
idea: seperate into large profit items (use above technique)
and small " " (pack afterwards)

We showed for knapsack
pseudo-poly. time alg. $\longrightarrow$ FPTAS.
Not known in general.
But converse true in general:
FPTAS $\Longrightarrow$ pseudo-poly. time alg.

Thm [Garey & Johnson '78]
If a problem has an FPTAS. (+ some technical assumptions)
then there's a pseudo-poly. time alg.

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idea of proof.
Suppose we have min-problem
and (technical assumption) obj. fn integer-valued
and bound OPT $< q(|I_{\text{unary}}|)$

q – some poly. size of input with numbers in unary.

Let A be FPTAS
run time $p(|I|, \frac{1}{\varepsilon})$ p – some poly.
$A(I) \le (1+\varepsilon) \cdot OPT(I)$

Pick ε small enough to get true OPT. Argue that this is pseudo-poly.

$$\varepsilon = \frac{1}{q(|I_{unary}|)}$$

$$A(I) \leq (1+\varepsilon)\cdot OPT(I) < OPT(I) + 1$$

because obj. is integer valued, $A(I)$ must give OPT.

Run time $p(|I|, \frac{1}{\varepsilon}) = \underbrace{p(|I|, q(|I_{unary}|)}_{\text{this is pseudo-poly. time.}}$
