Today’s Topics

• CN in Matrix form, Computational efficiency
• Convergence
  – consistency
  – stability
• Stability for explicit method: restriction on $\Delta \tau$
• Stability of the fully implicit method
Crank-Nicolson: Matrix Form

Let matrix $\hat{M}$ be defined so that (row $i$, $i \neq 0, m$)

$$\left[ \hat{M} V^n \right]_i = -\frac{\Delta \tau \alpha_i}{2} V^n_{i-1} + \frac{\Delta \tau (\alpha_i + \beta_i + r)}{2} V^n_i - \frac{\Delta \tau \beta_i}{2} V^n_{i+1}.$$  

Then, C-N timestepping (European option) can be written as

$$\left[ I + \hat{M} \right] V^{n+1} = \left[ I - \hat{M} \right] V^n$$

Applying CN to the boundary equation at $S = 0$, the first row of $\hat{M}$ corresponds to setting $\alpha_0 = \beta_0 = 0$.

What is the last row of $\hat{M}$?
Finite Difference for a LVF Model

Form of $\alpha_i^n$, $\beta_i^n$ depends on choice of central, forward, backward weighting

Central Weighting (difference)

$$
\alpha_{i,central}^n = \left[ \frac{(\sigma_i^n)^2 S_i^2}{(S_i - S_{i-1})(S_{i+1} - S_{i-1})} - \frac{rS_i}{S_{i+1} - S_{i-1}} \right]
$$

$$
\beta_{i,central}^n = \left[ \frac{(\sigma_i^n)^2 S_i^2}{(S_{i+1} - S_i)(S_{i+1} - S_{i-1})} + \frac{rS_i}{S_{i+1} - S_{i-1}} \right].
$$

(1)

$$
\sigma_i^n = \sigma(S_i, \tau^n)
$$
Finite Difference for a LVF Model

Forward difference

$$\alpha_{i,\text{forward}}^n = \frac{(\sigma_i^n)^2 S_i^2}{(S_i - S_{i-1})(S_{i+1} - S_{i-1})}$$

$$\beta_{i,\text{forward}}^n = \left[ \frac{(\sigma_i^n)^2 S_i^2}{(S_{i+1} - S_i)(S_{i+1} - S_{i-1})} + \frac{r S_i}{S_{i+1} - S_i} \right].$$  \hspace{1cm} (2)

Backward difference

$$\alpha_{i,\text{backward}}^n = \left[ \frac{(\sigma_i^n)^2 S_i^2}{(S_i - S_{i-1})(S_{i+1} - S_{i-1})} - \frac{r S_i}{S_i - S_{i-1}} \right]$$

$$\beta_{i,\text{backward}}^n = \frac{(\sigma_i^n)^2 S_i^2}{(S_{i+1} - S_i)(S_{i+1} - S_{i-1})}$$  \hspace{1cm} (3)

Note: discrete equations have same form, regardless of choice of central, forward, backward, only definition of $\alpha, \beta$ changes
Matrix Form of Equations for a LVF Model

Define the vectors

\[
V^{n+1} = \begin{bmatrix}
V_0^{n+1} \\
V_1^{n+1} \\
\vdots \\
V_m^{n+1}
\end{bmatrix}; \quad V^n = \begin{bmatrix}
V_0^n \\
V_1^n \\
\vdots \\
V_m^n
\end{bmatrix},
\]

Let \( M^n \) be the tridiagonal matrix with entries

\[
[M^n V^n]_i = -\Delta \tau \alpha_i^n V_{i-1}^n + \Delta \tau (\alpha_i^n + \beta_i^n + r) V_i^n - \Delta \tau \beta_i^n V_{i+1}^n
\]

Fully implicit timestepping for a European option can be written as

\[
[I + M^{n+1}] V^{n+1} = V^n.
\]
Crank-Nicolson: Matrix Form for a LVF Model

Let matrix $\hat{M}$ be defined so that (row $i$)

$$\left[\hat{M}^n V^n\right]_i = -\frac{\Delta \tau \alpha_i^n}{2} V_{i-1} + \frac{\Delta \tau (\alpha_i^n + \beta_i^n + r)}{2} V_i^n - \frac{\Delta \tau \beta_i^n}{2} V_{i+1}^n.$$

Then, C-N timestepping (European option) can be written as

$$\left[I + \hat{M}^{n+1}\right] V^{n+1} = \left[I - \hat{M}^n\right] V^n$$

What is $\sigma(S, t) = \sigma(S)$? How to achieve efficiency?
Solving Linear System

To solve numerically

\[ Ax = b \]

we can compute a permutation matrix \( P \), lower triangular matrix \( L \) and upper triangular matrix \( U \) such that

\[ PA = LU \]

Permutation is not necessary when \( A = (a_{ij}) \) is columnly diagonally dominant (which is the case for implicit/Crank-Nicolson method), i.e.,

\[ |a_{ii}| > \sum_{j \neq i} |a_{ji}|, \quad \forall i \]

Matlab functions \texttt{spdiags}, \texttt{lu}, \texttt{\textbackslash} the cost is roughly \( 8(m + 1) \) for such tridiagonal systems.

\textbf{Note}. It is important to input the coefficient matrix as a sparse matrix
Stability and Convergence

Error occurs due to

- Floating point arithmetic errors
- Errors due to approximating derivatives by finite differences
- Propagation error: error made at $\tau^n$ has impact on accuracy of the computed value at $\tau^{n+1}$. 
A FD method is **convergent** if

\[
\lim_{\Delta S, \Delta \tau \to 0} (V_i^n - V(S_i, \tau^n)) = 0.
\]

for all \(0 \leq i \leq N, 0 \leq j \leq m\)

Convergence of a FD method depends on

- local truncation error converges to zero as \(\Delta \tau, \Delta S \to 0\) and
- stability of the FD method.
Consider Black-Scholes equation,

\[ V_\tau - \mathcal{L}V = 0 \]

Consider a smooth function \( \psi \) such that \( \psi \) has bounded derivatives w.r.t. \((S, \tau)\).
DEFINITION 0.1 (Truncation Error) The Truncation Error (T.E.) of a discretization of the B.S. equation is

\[ T.E. = \left[ \{(\psi_\tau)_i^n - (\mathcal{L}\psi)_i^n \}_{\text{discrete}} \right] - \left[ \{(\psi_\tau) - (\mathcal{L}\psi)\}_i^n \right] \]
Let

\[ \Delta S = \max_i (S_{i+1} - S_i) \]
\[ \Delta \tau = \max_n (\tau^{n+1} - \tau^n) \]

**DEFINITION 0.2 (Consistency)** A discretization of the B.S. equation is consistent if

\text{Truncation Error} \to 0

As \( \Delta S, \Delta \tau \to 0 \)
Consistency Example

Consider heat equation:

\[ U_\tau = U_{SS} \quad (5) \]

which we discretize by

\[ \frac{U_{i}^{n+1} - U_{i}^{n}}{\Delta \tau} = \frac{U_{i-1}^{n} - 2U_{i}^{n} + U_{i-1}^{n}}{(\Delta S)^2} \quad (6) \]

Let \( \psi(S, \tau) \) be a smooth test function, then

\[ \frac{\psi_{i}^{n+1} - \psi_{i}^{n}}{\Delta \tau} = [\psi_\tau]_{i}^{n} + O(\Delta \tau) \]

\[ \frac{\psi_{i-1}^{n} - 2\psi_{i}^{n} + \psi_{i-1}^{n}}{(\Delta S)^2} = [\psi_{SS}]_{i}^{n} + O((\Delta S)^2) \quad (7) \]

\[ \Rightarrow \frac{\psi_{i}^{n+1} - \psi_{i}^{n}}{\Delta \tau} - \left( \frac{\psi_{i-1}^{n} - 2\psi_{i}^{n} + \psi_{i-1}^{n}}{(\Delta S)^2} \right) - [\psi_\tau - \psi_{SS}]_{i}^{n} = O(\Delta \tau) + O((\Delta S)^2) \quad (8) \]
A necessary condition for convergence is that the truncation error goes to zero as $\Delta \tau, \Delta S \to 0$. Such methods are said to be consistent.

Convergence rate:

- $O(\Delta \tau, (\Delta S)^2)$ for explicit/implicit method
- $O((\Delta \tau)^2, (\Delta S)^2)$ for Crank-Nicolson method

Heat equation example: (6) is a consistent discretization of (5).
A FD method is \textbf{stable} if the \textit{amplification} factor in error propagation is no larger than one.

Stability is a necessary and sufficient condition for convergence if the local truncation error goes to zero as $\Delta \tau, \Delta S \to 0$. 
Stability requirement can impose a condition on the choice of the stepsize.

For a heat equation, the explicit method is **stable** if

\[
0 < \frac{\Delta \tau}{(\Delta x)^2} \leq \frac{1}{2}
\]

<table>
<thead>
<tr>
<th>S</th>
<th>(\frac{\Delta \tau}{(\Delta x)^2}) = 0.5</th>
<th>(\frac{\Delta \tau}{(\Delta x)^2}) = 0.52</th>
<th>exact</th>
</tr>
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<td>0.4419</td>
<td>625.0347</td>
<td>0.4420</td>
</tr>
<tr>
<td>11</td>
<td>0.1607</td>
<td>-457.3122</td>
<td>0.1606</td>
</tr>
</tbody>
</table>

WDH: puts with \(K = 10\), \(r = 5\%\), \(\sigma = 20\%\), \(T = 1/2\).
Consider the explicit method for the Black-Scholes equation

\[ V_{i}^{n+1} = V_{i}^{n} (1 - (\alpha_i + \beta_i + r)\Delta\tau) + \alpha_i \Delta\tau V_{i-1}^{n} + \beta_i \Delta\tau V_{i+1}^{n}. \]

- **Basic test for stability**: require that the discrete value of the option is *finite* for fixed \( T, S_{\text{max}} \) as \( \Delta \tau, \Delta S \) goes to zero (Thus the number of steps goes to \(+\infty\)).
Requirement for stability is simply that for a finite computational domain \((S \in [0, S_{max}], \tau \in [0, T])\), \(V_{i}^{n}\) remains bounded as \(n \to \infty\).

At the boundary \(S_{max}\)

\[ V_{m+1}^{n} = V_{m}^{n} \]
We will assume that upstream or central weighting is selected so that

\begin{align*}
\alpha_i & \geq 0 \\
\beta_i & \geq 0.
\end{align*}

\( (9) \)

From the FD equation and nonnegativity of \( \alpha_i \) and \( \beta_i \), for explicit method, we obtain

\[ |V_i^{n+1}| \leq |V_i^n (1 - (\alpha_i + \beta_i + r)\Delta \tau)| + |V_{i-1}^n|\alpha_i \Delta \tau + |V_{i+1}^n|\beta_i \Delta \tau \]
Now assume that

\[ 1 - (\alpha_i + \beta_i + r) \Delta \tau \geq 0 \quad ; \quad \forall i, \]

Then

\[ |V_{i}^{n+1}| \leq |V_{i}^{n}| (1 - (\alpha_i + \beta_i + r) \Delta \tau) + |V_{i-1}^{n}| \alpha_i \Delta \tau + |V_{i+1}^{n}| \beta_i \Delta \tau. \]

Let

\[ \|V^n\|_{\infty} = \max_i |V_i^n|. \]

Then we have

\[ |V_{i}^{n+1}| \leq \|V^n\|_{\infty} (1 - r \Delta \tau) \]

\[ \leq \|V^n\|_{\infty}, \quad \text{for any } i. \]
Note that the above clearly holds for $S = 0$. Hence

$$\|V^{n+1}\|_\infty \leq \|V^n\|_\infty.$$  \hfill (11)

Now, assuming that $\|V^0\|$ is bounded equation (11) says that the discrete solution is bounded as $n \to \infty$.

In other words, the discretization is *stable.*
The key stability condition is equation (10), which can be rewritten as

\[ \Delta \tau \leq \frac{1}{\alpha_i + \beta_i + r} ; \quad \forall i, \quad (12) \]

or

\[ \Delta \tau \leq \min_i \left( \frac{1}{\alpha_i + \beta_i + r} \right). \]

Note that the above argument establishes that condition (12) is sufficient for stability. It can also be shown that condition (12) is a necessary condition.
Stability for the Explicit Method

For the explicit method, stability restriction on the stepsize limits computational efficiency.

- The time stepsize limitation can be severe if refinement discretization is used near discontinuity.

- A binomial or tri-nomial method is algebraically identical to an explicit finite difference scheme, with a maximum time stepsize for stability.
Stability of Implicit Methods

Implicit

\[ V_i^{n+1} = V_i^n + \Delta \tau \left[ \alpha_i (V_{i-1}^{n+1} - V_i^{n+1}) + \beta_i (V_{i+1}^{n+1} - V_i^{n+1}) - rV_i^{n+1} \right] \]

where \( \alpha_i, \beta_i \) are selected from either central or upstream weighting to ensure \( \alpha_i \geq 0; \beta_i \geq 0 \).

We can similarly analyze the stability of the fully implicit method.
We can write equation the equation equivalently as

\[ V_i^{n+1} [1 + \Delta\tau(r + \alpha_i + \beta_i)] = V_i^n + \Delta\tau \alpha_i V_{i-1}^{n+1} + \Delta\tau \beta_i V_{i+1}^{n+1}. \]

Noting that \( \alpha_i, \beta_i, r \) are all nonnegative, it follows that

\[ |V_i^{n+1}| [1 + \Delta\tau(r + \alpha_i + \beta_i)] \leq |V_i^n| + |V_{i-1}^{n+1}| \Delta\tau \alpha_i + |V_{i+1}^{n+1}| \Delta\tau \beta_i. \]

Thus

\[ |V_i^{n+1}| [1 + \Delta\tau(r + \alpha_i + \beta_i)] \leq \|V^n\|_{\infty} + \Delta\tau(\alpha_i + \beta_i)\|V^{n+1}\|_{\infty}. \]
Since this is true for all $i$, letting $|V_{i*}^{n+1}| = \|V^{n+1}\|_\infty$, we have

$$\|V^{n+1}\|_\infty [1 + \Delta \tau (r + \alpha_i* + \beta_i*)]$$

$$\leq \|V^n\|_\infty + \Delta \tau (\alpha_i* + \beta_i*) \|V^{n+1}\|_\infty$$

$$\Rightarrow \|V^{n+1}\|_\infty \leq \frac{\|V^n\|_\infty}{1 + r \Delta \tau} \leq \|V^n\|_\infty.$$

Therefore a fully implicit method is unconditionally stable, i.e., there is no restriction on timestep size.