Today’s Topics

- Option model calibration
- Newton and Levenberg-Marquardt methods
- Matlab \texttt{lsqnonlin, fmincon}
- CVaR optimization
A Newton Step

Assume that $x_c$ is the current approximation.

A Newton step for $\min \frac{1}{2} \| F(x) \|_2^2$ is

$$x_+ = x_c - (J(x_c)^T J(x_c) + S(x_c))^{-1} J(x_c)^T F(x_c)$$

where $S(x) = \sum_{i=1}^{m} \nabla^2 F_i(x) F_i(x)$. 
Jacobian: Finite Difference

Consider a single function $g(z)$ of one variable $z \in \mathbb{R}$.

$$g'(z) = \frac{g(z + \Delta z) - g(z)}{\Delta z} + O(\Delta z)$$

Suppose that the function is already evaluated at $z = z_c$.

The derivative can be approximated by one additional function evaluation $g(z + \Delta z)$.

$\Rightarrow$ Choose $\Delta z \approx \sqrt{\text{eps}}$. 
Approximate Jacobian Using FD

Assume $F(x_c)$ has been computed.

Let $\Delta \approx \sqrt{\epsilon}$ and $e_i = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$:

The $m$-by-$n$ Jacobian matrix can be approximated by:

```
for j = 1 : m
    for i = 1 : n
        \frac{\partial F_j}{\partial x_i} \approx \frac{F_j(x_c + \Delta \cdot e_i) - F_j(x_c)}{\Delta}
    end
end
```

Each $F_j(x_c + \Delta \cdot e_i)$ requires a PDE solve
The Levenberg-Marquardt method

But Newton methods converge only locally.

Trust region methods: Determine $x_+$ via solving a trust region subproblem: for some $\Delta_c \geq 0$

$$
\min_{x_+} \|F(x_c) + J(x_c)(x_+ - x_c)\|_2^2
$$

subject to $\|x - x_c\|_2 \leq \Delta_c$

Trust region size $\Delta_k$ is updated adaptively.
It can be shown that the solution to the trust region subproblem has the form

\[ x_+ = x_c - (J(x_c)^T J(x_c) + \mu_c I)^{-1} J(x_c)^T F(x_c) \]

for some \( \mu_c > 0 \).
Minimizing Nonlinear Least Squares: $\min_x \frac{1}{2} \| F(x) \|^2$

- Calibration problem can have multiple local minimizers
  - Choosing a good starting point can help locating a global minimizer

- Levenberg-Marquardt uses only the Jacobian matrix; only partial Hessian information is used
  - Iterations can progress slowly in a very nonlinear region with large residuals
  - Choosing a good starting point can help

- Alternative optimization methods use the exact/approximate Hessian info (expensive)
  - Automatic differentiation can be used
Matlab

The matlab function \texttt{lsqnonlin} solves a nonlinear least squares problem.

\[
x = \text{lsqnonlin} (\text{fun}, x_0, lb, ub, \text{options})
\]

\[
[x, \text{resnorm, residual, exitflag}] = \text{lsqnonlin}(...)
\]

\[
[x, \text{resnorm, residual, exitflag, output}] = \text{lsqnonlin}(...)
\]

Setting options for \texttt{lsqnonlin}:

\[
\text{options} = \text{optimset} (\text{Jacobian, on, Algorithm, levenberg-marquardt, Display, iter, MaxIter, 50})
\]
function [F,J] = myfun(x)
F = ... % Objective function values at x
if nargout > 1 % Two output arguments
    J = ... % Jacobian of the function evaluated at x
End

For more details, see doc lsqnonlin in matlab.
Matlab: Minimizing Nonlinear Function Under Constraints

\[
x = \textbf{fmincon}(fun, x0, A, b, Aeq, beq, lb, ub)
\]

\[
\text{options=} \textbf{optimset}('\text{'Algorithm'},'\text{trust-region-reflective'}');
\]

Computing CVaR Optimal Portfolios

**Task in a hedge fund:** $n$ financial assets (including options) with random loss $L \in \mathbb{R}^n$ in a time horizon $[0, t]$, e.g., a month. Assume that the loss distribution is known.

**Example:** different options on two stocks $S^A$ and $S^B$ and $L = V_0 - V_t$

Denote $x \in \mathbb{R}^n$ as percentage holdings in these $n$ assets. Portfolio loss is $L^T x$

**Objectives:** choose $x$ to

- minimizer risk
- maximize expected return

**Notation.** The subsequent discussion defines VaR/CVaR wrt loss, which is $-P&L$. 
Computing CVaR Optimal Portfolios

Assume that the assets have continuous loss distribution. Then the portfolio loss has a continuous CDF $\Psi(x, \cdot)$

**Note.** $x$ is the asset percentage holdings
Portfolio CVaR Optimization

Denote \( z^+ = \max(z, 0) \), portfolio loss \( f(x, L) = x^T L \). Consider:

\[
F_\beta(x, \alpha) = \alpha + (1 - \beta)^{-1} \mathbb{E} \left[ (f(x, L) - \alpha)^+ \right]
\]

Rockafellar and Uryasev [2002]:

\[
\text{CVaR}_\beta (f(x, L)) = \min_{\alpha \in \mathbb{R}} F_\beta(x, \alpha)
\]

(See later for convex optimization discussion and Q1, Assignment 4)

\[
\min_{x \in \mathcal{X}} \text{CVaR}_\beta (f(x, L)) \iff \min_{x \in \mathcal{X}} \min_{\alpha \in \mathbb{R}} F_\beta(x, \alpha) \\
\iff \min_{(x, \alpha) \in \mathcal{X} \times \mathbb{R}} F_\beta(x, \alpha)
\]
Computing a Minimum CVaR Portfolio

Minimum CVaR Risk Portfolio:

\[
\min_{(x, \alpha) \in \mathcal{X} \times \mathbb{R}} F_\beta(x, \alpha)
\]

(1)

- The probability density for the portfolio loss function \( f(x, L) \) is often not explicitly given.

- We assume that discrete (equally likely) samples \( \{(L)_i\}_{i=1}^m \) available, \( m \) is typically large, e.g., \( m = 10^5 \).

- If \( L \) is a portfolio of stocks and options on the stocks, how will you compute \( \{(L)_i\}_{i=1}^m \)?
Let
\[ \mathbb{E}(L) \approx \bar{L} = \frac{1}{m} \sum_{i=1}^{m} (L)_i \]

By definition of CVaR, for any given portfolio \( x \),
\[ \text{CVaR}_\beta \left( L^T x \right) = \min_\alpha \alpha + \frac{1}{1 - \beta} \mathbb{E} \left( \left( L^T x - \alpha \right)^+ \right). \]

Replacing expectation by sample mean:
\[ \text{CVaR}^S_\beta \left( L^T x \right) \approx \min_\alpha \alpha + \frac{1}{(1 - \beta)m} \sum_{i=1}^{m} \left( \left( (L)_i x - \alpha \right)^+ \right) \]
• Let \( X \) denote the set of portfolios with avg loss \( \bar{\ell} \), \( e^T x = 1 \) and \( x \geq 0 \).

• Minimum CVaR portfolio (1) can be computed by

\[
\min_{(x, \alpha)} \alpha + \frac{1}{m(1-\beta)} \sum_{i=1}^{m} ((L_i^T x - \alpha)^+ \\
\text{s.t.} \begin{cases} 
\bar{L}^T x = \bar{\ell} \\
e^T x = 1 \\
x \geq 0 
\end{cases}
\]
Formulation: Constraining CVaR risk

In practice, often a risk budget is given. We want the portfolio $x$ with maximum expected return subject to an upper bound on the risk based CVaR.

A CVaR constrained portfolio optimization problem

$$\min_{x} \quad x^T \mathbf{E}(\mathbf{L})$$

subject to

$$e^T x = 1, \quad x \geq 0$$

$$\text{CVaR}_\beta \left( \mathbf{L}^T x \right) \leq \Lambda$$

where $\Lambda$ is a given constant.
Solving CVaR Constrained Problems as LP

Thus the sample CVaR optimization problem for (3) is

$$\min_x \bar{L}^T x$$

subject to

$$e^T x = 1, \quad x \geq 0$$

$$\left( \min_{\alpha} \alpha + \frac{1}{(1 - \beta)m} \sum_{i=1}^{m} ((L_i^T x - \alpha)^+) \right) \leq \Lambda$$

$\bar{L}$: sample mean loss, $\approx \mathbb{E}(L)$
Let
\[ z_i = \left( (L)_i^T x - \alpha \right)^+ \]
then
\[ z_i \geq 0 \quad \text{and} \quad z_i \geq (L)_i^T x - \alpha \]
It can be easily verified that $\alpha, x$ satisfy
\[
\left( \min_{\alpha} \alpha + \frac{1}{(1 - \beta)m} \sum_{i=1}^{m} \left( (L)_i^T x - \alpha^+ \right) \right) \leq \Lambda
\]
if and only if there exits $z$ such that $\alpha, x, z$ satisfy
\[
\alpha + \frac{1}{(1 - \beta)m} \sum_{i=1}^{m} z_i \leq \Lambda
\]
\[
z_i - (L)_i^T x + \alpha \geq 0, \quad i = 1, \cdots, m
\]
\[
z \geq 0
\]
Hence CVaR constrained problem (3) is equivalent to (5) below:

\[
\begin{align*}
\min_{x,z,\alpha} & \quad \bar{L}^T x \\
\text{subject to} & \quad e^T x = 1, \quad x \geq 0 \\
& \quad \alpha + \frac{1}{(1 - \beta)m} \sum_{i=1}^{m} z_i \leq \Lambda \\
& \quad z_i - (L)^T_i x + \alpha \geq 0, \quad i = 1, \cdots, m \\
& \quad z \geq 0
\end{align*}
\]
Solving LP

LP has

- $O(m + n)$ variables
- $O(m + n)$ constraints
- LP software, e.g., MOSEK, CVX, & CPLEX, are efficient and robust package

In Matlab,

$$z = \text{linprog} \left( f, A, b, A_{eq}, b_{eq}, l, u \right)$$

$$\min_z \quad f^T z$$
subject to

$$Az \leq b$$
$$A_{eq}z = b_{eq}$$
$$l \leq z \leq u$$
What if we minimize VaR?

$\text{VaR}_{0.95}$ and $\text{CVaR}_{0.95}$ with respect to asset 1 position for a two-asset portfolio
Minimum Risk Portfolio

- Minimizing VaR is a nonconvex problem (more later): there can be many local minimizers and computing a global minimizer is very difficult.

- Minimizing CVaR is a convex problem and a minimizer can be computed efficiently.

Subsequently we examine this more formally