Today’s Topics

- Convex Sets
- Convex functions
- How to establish convexity?
- Operations that maintain convexity
- Convex optimization problems

\[\text{Note: basic discussion on convexity and optimality follows Convex Optimization, Boyd and Vandenberghe, see https://stanford.edu/boyd/cvxbook}\]
Convex Set

A set $D$ is **convex** if the line segment between any two points in $D$ lies in $D$, i.e., if for any $x_1, x_2, \in D$ and any $0 \leq \theta \leq 1$, we have

$$\theta x_1 + (1 - \theta)x_2 \in D$$

Examples: $\mathbb{R}^n$, $x > 0$, $\|x\|_2 \leq 1$ are convex

What about $D = [1, 2] \cup [4, 5]$?
Convex Function

$f(\cdot) : \mathbb{R}^n \to \mathbb{R}$ is convex if $dom f$ is a convex set and

$$f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y)$$

for all $x, y \in dom f, 0 \leq \theta \leq 1$.

$f$ is concave if $-f$ is convex.
\( f(\cdot) : \mathbb{R}^n \rightarrow R \) is log-concave (log-convex) if \( f > 0 \) for all \( x \in \text{dom} f \) and \( \log f \) is concave (convex).

\( f \) is \textbf{strictly convex} if \( \text{dom} f \) is convex and

\[
f(\theta x + (1 - \theta)y) < \theta f(x) + (1 - \theta)f(y)
\]

for all \( x, y \in \text{dom} f, x \neq y, 0 < \theta < 1.\)
Convex Functions

How do we establish convexity?

Restricting to line: $f(\cdot)$ is convex $\iff$ for all $x \in \text{dom} f$ and for all $v$
$g(t) = f(x + tv)$ is convex on its domain $\{t : x + tv \in \text{dom} f\}$
First-order condition

$f(\cdot)$ is differentiable if $domf$ is open and the gradient
\[ \nabla f(x) = \left( \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots, \frac{\partial f}{\partial x_n} \right) \]
exists at each $x \in domf$.

Differentiable $f$ with a convex domain is convex iff

\[ f(y) \geq f(x) + \nabla f(x)^T(y - x) \quad \text{for all} \quad x, y \in domf \]

$\Rightarrow$ first order approximation is a global underestimator of $f(\cdot)$
Proof of First-order condition

Proof: Consider the case $n = 1$: We prove that a differentiable function $f : \mathbb{R} \to \mathbb{R}$ is convex iff

$$f(y) \geq f(x) + f'(x)(y - x) \quad (1)$$

for all $x, y \in \text{dom } f$.

$\Rightarrow$: Assume that $f$ is convex and $x, y \in \text{dom } f$. Since $\text{dom } f$ is convex (i.e., an interval), for all $0 < t < 1, x + t(y - x) \in \text{dom } f$, and by convexity of $f$,

$$f(x + t(y - x)) \leq (1 - t)f(x) + tf(y).$$

If we divide both sides by $t$, we obtain

$$f(y) \geq f(x) + \frac{f(x + t(y - x)) - f(x)}{t}$$

and taking the limit as $t \to 0$ yields (1).
\( \Leftarrow \): assume the function satisfies (1) for all \( x, y \in \text{dom} f \) (which is an interval). Choose any \( x \neq y \), and \( 0 \leq \theta \leq 1 \), and let \( z = \theta x + (1 - \theta)y \). Applying (1) twice yields

\[
f(x) \geq f(z) + f'(z)(x - z), \quad f(y) \geq f(z) + f'(z)(y - z).
\]

Multiplying the first inequality by \( \theta \), the second by \( 1 - \theta \), and adding them yields

\[
\theta f(x) + (1 - \theta)f(y) \geq f(z), \text{i.e., } f \text{ is convex}
\]
Second-order conditions

$f(\cdot)$ is twice differentiable if $dom f$ is open and the Hessian $\nabla^2 f(x)$, $n$-by-$n$ symmetric matrix

$$(\nabla^2 f(x))_{ij} = \frac{\partial^2 f(x)}{\partial x_i \partial x_j}, \quad i, j = 1, \ldots, n,$$

exists at each $x \in dom f$

For twice differentiable $f(\cdot)$ with a convex domain $dom f$, $f$ is convex if and only if

$$\nabla^2 f(x) \succeq 0 \quad \text{for all} \ x \in dom f$$

if $\nabla^2 f(x) \succ 0$ for all $x \in dom f$, then $f$ is strictly convex
1-D Examples

convex:

• \( ax + b \) on \( R \), for any \( a, b \in R \)
• \( e^{ax} \), for any \( a \in R \)
• \( x^\alpha \) when \( x > 0 \), for \( \alpha \geq 1 \)
• \(|x|^p\) on \( R \), for \( p \geq 1 \)

concave:

• \( ax + b \) on \( R \), for any \( a, b \in R \)
• \( x^\alpha \) when \( x > 0 \), for \( 0 \leq \alpha \leq 1 \)
• \( \log(x) \) when \( x > 0 \)
Examples on $\mathbb{R}^n$

- affine function $f(x) = a^T x + b$

- norms: $\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$ for $p \geq 1$; $\|x\|_\infty = \max_k |x_k|$
Positively weighted sum and composition

• $\sum_{i=1}^{m} \alpha_i f_i(x)$ is convex if $\{f_i(x)\}$ are convex and $\alpha_i \geq 0$ (extends to infinite sums, integrals)

• $f(Ax + b)$ is convex if $f(\cdot)$ is convex.

Example: norm of affine function: $f(x) = \|Ax + b\|$
Maximum and Supremum

- If \( f_1(x), \ldots, f_m(x) \) are convex, then
  \[
  f(x) = \max\{f_1(x), \ldots, f_m(x)\}
  \]
  is convex.

  Example: \( f(x) = \max_i (a_i^T x - b_i) \)

- If \( f(x, y) \) is convex in \( x \) for each \( y \in D_y \), then
  \[
  g(x) = \sup_{y \in D_y} f(x, y)
  \]
  is convex.

  Example: \( f(x) = \max_{y \in D_y} \|x - y\| \)
Convex Optimization

\[
\min_x f(x) \\
\text{subject to} \quad g_i(x) \leq 0, \quad i = 1, \ldots, m \\
q_j(x) = 0, \quad j = 1, \ldots, l
\]

- When \( f(x), g_i(x) \) are all convex and \( q_j(x) \) is linear, the problem is convex optimization

- Maximizing a concave function over convex set is
convex opt problem

This optimization problem will be called \textit{primal problem}.

A point satisfying the constraints is said to be \textit{feasible}. 
Lagrangian and Optimality

• **Lagrangian:** $L : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^l \rightarrow \mathbb{R}$,

  \[ L(x, \lambda, \nu) \overset{\text{def}}{=} f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) + \sum_{j=1}^{l} \nu_j q_j(x) \]

  • $\lambda_i$: Lagrange multiplier associated with $g_i(x) \leq 0$

  • $\nu_j$: Lagrange multiplier associated with $q_j(x) = 0$
Lagrangian is central in optimization theory and algorithms (e.g., recent revival of ADMM is important for solving big data problem)