CS 476/676: Assignment 1
Winter 2020,
Version January 20, 2020

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Lecture Times: MW 2:30-3:50 E2 1732
Yuying Li, OH: Thursday 10:00-11:00 DC3623
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OH (Chendi, DC3594): Tuesday, 21 Jan, 4-6pm Wednesday, 29 Jan, 4-5pm
OH (Chendi, DC3594): Friday, 31 Jan 5-6pm Monday, 3 Feb, 4-5pm

Course Web Site: http://www.student.cs.uwaterloo.ca/~cs476
My Web Site: http://www.cs.uwaterloo.ca/~yuying

Due February 5, 2020 in class

IMPORTANT: In this and in future assignments, most of the marks for programming questions are allocated for explanations of algorithms (e.g. pseudo-code) and discussion of results. All coding should be done in Matlab. All the plots should be appropriately labeled. By default, you should submit listings of all Matlab code used in your assignment. Be sure to document (i.e. add liberal comments) your code. The TA will take off marks for poor documentation.

1. (14 marks) (Imply a Binomial Lattice from Option Prices)

Assume that the ABC stock pays no dividend and is currently priced at $S_0 = 18$. Assume that, at the expiry time $T > 0$, the stock price goes up to $uS_0$ with probability $0 < p < 1$ and down to $dS_0$ with probability $1 - p$, where $p = 0.5$. We know that $d < 1 < u$ but do not know $d$ or $u$. Assume that there is no arbitrage and the interest rate is zero. Consider the following three options with the same expiry $T$ on the ABC stock.

Assume that a European put option with strike price $18$ is priced at $4$ while another European put option with strike price $14$ is priced $rac{4}{3}$.

(a) What is the fair value of a European call option with a strike price of $15$ and expiry $T$?

(b) How many units of the underlying is required at $t = 0$ to hedge a short position in this call? Explain your answer.

(c) Using the actual probability $p$, what is the expected option payoff for the European call in (a)? What is wrong with pricing this call option at this expected payoff value? If this European call option is priced at the expected payoff using $p$, how can you construct an arbitrage?

2. (10 marks) (Binomial Option Values)

Under the lognormal Black-Scholes model,

$$dS = \mu S dt + \sigma S dZ,$$

(1)

given the stock price $S_n$ at time $t_n$ and $\Delta t = t_{n+1} - t_n$, it can be shown that

$$S(t_{n+1}) = S(t_n)e^{(\mu - \frac{1}{2} \sigma^2)\Delta t + \sigma \sqrt{\Delta t}}, \quad \phi \sim \mathcal{N}(0,1)$$

(2)

Assume that a binomial lattice satisfies

$$pu + (1 - p)d = e^{\mu \Delta t}, \quad pu^2 + (1 - p)d^2 = e^{(2\mu + \sigma^2)\Delta t}$$

(3)

Show that a binomial model satisfying (3) converges to the lognormal Black-Scholes model (1) as $\Delta t \to 0$. Hint: compute explicitly $E(S(t_{n+1}))$ and $E((S_{t_{n+1}})^2)$ and compare mean and variance to those from the binomial model.
3 (14 marks) (Lattice Methods).

Assume that the continuously compounding interest rate is the constant \( r > 0 \). Assume that the market price follows the following binomial lattice model: at \( t_n = n\Delta t \), the stock price is \( S^n_j \) and the stock price at \( t_{n+1} = (n+1)\Delta t \) equals

\[
\begin{align*}
S^n_{j+1} &= uS^n_j, \quad \text{with probability } p \\
S^n_{j-1} &= dS^n_j, \quad \text{with probability } 1-p
\end{align*}
\]

where \( 0 < p < 1 \) and \( d \leq e^{r\Delta t} \leq u \).

Consider two European put options with the same expiry \( T \) but different strike \( K \) and \( \tilde{K} \), \( K > \tilde{K} \), and denote the corresponding put values at time \( t \), \( 0 \leq t \leq T \), by \( P^n_t \) and \( \tilde{P}^n_t \) respectively. Let \( P^n_j \) and \( \tilde{P}^n_j \) be the corresponding values of European puts obtained using a no-arbitrage lattice at node \((j,n)\) and expiry time \( T = N\Delta t \). Prove that, for \( 0 \leq n \leq N \) and \( 0 \leq j \leq n \),

\[
P^n_j - \tilde{P}^n_j \leq (K - \tilde{K})e^{-r(N-n)\Delta t}
\]

4. (4 marks) (Binomial Lattice)

Consider the drifting lattice \( S^1_{j+1} = uS^1_j \); \( S^1_{j-1} = dS^1_j \) where

\[
\begin{align*}
u &= \exp\left[\sigma\sqrt{\Delta t} + (r - \sigma^2/2)\Delta t\right] \\
d &= \exp\left[-\sigma\sqrt{\Delta t} + (r - \sigma^2/2)\Delta t\right]
\end{align*}
\]

and the risk neutral probability

\[
q^* = \frac{e^{r\Delta t} - d}{u - d}
\]

(a) Show that \( q^* \) converges to \( \frac{1}{2} \) as \( \Delta t \to 0 \).

(b) What continuous distribution does the binomial lattice (5) converge to? Hint: use Question 2.

5. (4 marks) (Properties of a Standard Brownian Motion)

Assume that a stochastic process \( S_t \) satisfies

\[
dS_t = \mu dt + \sigma dZ_t,
\]

where \( \mu > 0 \) and \( \sigma > 0 \) are constants. What is the stochastic differential equation satisfied by \( S(t)^2 \)? Note that \( S(t)^2 \) denotes the square of \( S(t) \). Show that

\[
\int_0^T S(t)dS(t) = \frac{S(T)^2 - S(0)^2}{2} - \left(\frac{T}{2}\right)\sigma^2
\]

6. (14 marks) (Binomial Lattice: Programming Questions)

Consider the drifting lattice \( S^1_{j+1} = uS^1_j \); \( S^1_{j-1} = dS^1_j \) where

\[
\begin{align*}
u &= \exp\left[\sigma\sqrt{\Delta t} + (r - \sigma^2/2)\Delta t\right] \\
d &= \exp\left[-\sigma\sqrt{\Delta t} + (r - \sigma^2/2)\Delta t\right] \\
q^* &= \frac{e^{r\Delta t} - d}{u - d}
\end{align*}
\]
Table 1: Some typical option parameters

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma$</td>
<td>25%</td>
</tr>
<tr>
<td>$r$</td>
<td>2.5%</td>
</tr>
<tr>
<td>Time to expiry</td>
<td>1 years</td>
</tr>
<tr>
<td>Initial asset price $S(0)$</td>
<td>$100</td>
</tr>
</tbody>
</table>

Implement European option pricing under a discrete dividend for the binomial lattice. Your code should take only $O(N)$ storage NOT $O(N^2)$, $N = T/\Delta t$. This can be done efficiently if you store the payoff in an array of size $O(N)$ and index it appropriately.

Assume that the underlying stock pays dividend $D = \rho S(t)$ at time $t = t_d = \frac{T}{2}$ when the stock price is $S(t)$ (the dividend rate is a constant $\rho > 0$). Note that, when the stock pays dividend at $t$, the stock price at $t^+$ becomes $S(t^-) - D$, where $t^-$ and $t^+$ denote immediately before and after the time of stock dividend payment respectively. Under no arbitrage, the option value immediately before dividend payment and immediately after dividend payment should remain the same. In §5.5.2 of the course notes, the case when dividend is a fixed amount is discussed. Modify the binomial lattice call and put option pricing to handle the dividend payment case in this question using interpolation similar to the technique described in §5.5.2.

(a) Using data in Table 1 and your binomial pricing code but assuming $\rho = 0$ (i.e. no dividend), test your code for standard European call and put by comparing your results with the exact solutions from the Matlab function `blsprice`. Show tables with the initial option value $V(S(0),0)$ (both at-the-money puts and calls, i.e., $K = S(0)$) as a function of $\Delta t$. Start off with a timestep $(\Delta t)^0 = .05$, and show the option value for $(\Delta t)^0/2, (\Delta t)^0/4, \ldots$. You should see your results converging to the `blsprice` value. Your tables should look like Table 2.

Table 2: Convergence Test

<table>
<thead>
<tr>
<th>$\Delta t$</th>
<th>Value</th>
<th>Change</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>.05</td>
<td>$V_1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>.025</td>
<td>$V_2$</td>
<td>$V_2 - V_1$</td>
<td></td>
</tr>
<tr>
<td>.0125</td>
<td>$V_3$</td>
<td>$V_3 - V_2$</td>
<td>$\frac{V_3-V_1}{V_2-V_1}$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

Let $V_{0}^{\text{exact}}$ be the exact initial price $V(S(0),0)$ (from the solution to the Black-Scholes PDE), and $V_{0}^{\text{tree}}(\Delta t)$ be the price from a lattice pricer with the time step $\Delta t$. If convergence rate is linear,

$$V_{0}^{\text{tree}}(\Delta t) = V_{0}^{\text{exact}} + \alpha \Delta t + o(\Delta t)$$  \hspace{1cm} (6)

where $\alpha$ is a constant independent of $\Delta t$, then the ratio

$$\lim_{\Delta t \to 0} \frac{V_{0}^{\text{tree}}((\Delta t)/2) - V_{0}^{\text{tree}}(\Delta t)}{V_{0}^{\text{tree}}((\Delta t)/4) - V_{0}^{\text{tree}}((\Delta t)/2)}$$  \hspace{1cm} (7)

would approach 2. If convergence rate is quadratic, then

$$V_{0}^{\text{tree}}(\Delta t) = V_{0}^{\text{exact}} + \alpha (\Delta t)^2 + o((\Delta t)^2)$$  \hspace{1cm} (8)

where $\alpha$ is some constant independent of $\Delta t$. Does your convergence table indicate a linear or quadratic convergence rate? Explain.

(b) Generate tables of fair values of the same at-the-money call and put options using $\Delta t = 0.01$, assuming dividend yield $\rho = 0.4\%, 8\%$ respectively. How do call and put values change with the dividend?
7. **(12 marks: total)** (Monte Carlo)

A European down-and-out call barrier option pays out the call payoff at the expiry $T$ when the underlying price $S_t$ never hits the barrier $B < S_0$ in $t \in [0, T]$. Otherwise it pays nothing.

Assume that the asset price $S$ evolves, in the risk neutral world, as follows:

$$dS = rSdt + \sigma SDZ$$

where $r$ is the risk free return, $\sigma$ is the volatility, and $dZ$ is the increment of a standard Brownian motion.

(a). **(4 marks)** Consider a down-and-out call option with the expiry $T = 1$ year. The initial asset price and the strike are $S_0 = K = 100$, the down barrier is $B = 85$. The analytic formula for pricing this barrier option is

$$V(S, t) = S \left( N(d_1) - (B/S)^{1+2r/\sigma^2} (1 - N(d_8)) \right) - Ke^{-r(T-t)} \left( N(d_2) - (B/S)^{-1+2r/\sigma^2} (1 - N(d_7)) \right)$$

where

$$d_1 = \frac{\log(S/B) + (r + \frac{1}{2} \sigma^2)(T-t)}{\sigma \sqrt{T-t}}$$
$$d_2 = d_1 - \sigma \sqrt{T-t}$$
$$d_3 = \frac{\log(S/B) + (r + \frac{1}{2} \sigma^2)(T-t)}{\sigma \sqrt{T-t}}$$
$$d_4 = d_3 - \sigma \sqrt{T-t}$$
$$d_5 = \frac{\log(S/B) - (r - \frac{1}{2} \sigma^2)(T-t)}{\sigma \sqrt{T-t}}$$
$$d_6 = d_5 - \sigma \sqrt{T-t}$$
$$d_7 = \frac{\log(S/B) - (r - \frac{1}{2} \sigma^2)(T-t)}{\sigma \sqrt{T-t}}$$
$$d_8 = d_7 - \sigma \sqrt{T-t}$$

and $N(d)$ is the cumulative distribution function for normal distribution, which is `normcdf` in Matlab. Compute and plot the initial down-and-out call option for $S = 90 : 2 : 110$. Other parameters can be found in Table 1.

(b). **(8 marks)** Let

$$N = \frac{T}{\Delta t}$$
$$S(n\Delta t) = S_n$$

Then, given an initial price $S(0)$, $M$ realizations of the path of a risky asset are generated using the algorithm

$$S_{n+1} = S_n e^{(r - \frac{1}{2} \sigma^2)\Delta t + \sigma \phi_n \sqrt{\Delta t}}$$

where $\{\phi_n\}$ are independent standard normals.

If the $m$th simulation value of $S_N = S(T)$ is denoted by $(S_N)^m$, then an approximate initial value of the option is given by (assuming initial price is $S(0)$)

$$\hat{V}(S(0), 0) = e^{-rT} \frac{\sum_{m=0}^{M} \text{Payoff}((S_N)^m)}{M}$$

For down-and-out call

$$\text{Payoff}(S_N) = \begin{cases} 
\max(S_T - K, 0), & \text{if } S_t > B, \text{ for } 0 \leq t \leq T \\
0, & \text{otherwise.}
\end{cases}$$

- The price simulation using (11) has no time discretization error. Does the error in the computed value $\hat{V}(S(0), 0)$ depend on the time discretization? Explain.
- Code up this MC algorithm in Matlab to determine the initial value of the European down-and-out call. Vectorize your code for efficiency.
- For a fixed $\Delta t$ plot the MC computed option values versus the number of simulations for a number of values of $M$. Repeat this computation for different $\Delta t$. Show the exact price computed from (a) on the plot. Explain what you see. You might start out using a timestep of 5 days (assuming 250 trading days in a year) with $M = 1000$. 


8. **(Graduate Student Question) (10 marks)**

   Explain why the Monte Carlo method in Problem 6 is slow in obtaining an accurate barrier option value. Read the recent short paper by Nouri et al, *Implementation of the modified Monte Carlo simulation for evaluate the Barrier option prices*, 2017, which can be downloaded from the course website. Read the paper, implement the computation method in the paper, and produce a table, similar to Table 1 in the paper by Nouri et al, but using the parameters in Q7 in this assignment.

   Comment on your observations of the computation results. Compare and discuss the improvement of the results compared to your implementation in Problem 7.