An Introduction to Computational Finance

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Some Facts?

• Populations in the developed world are living longer
  → More people will need to live on invested capital in their retirement years.

• Conventional wisdom
  → Stocks are good investments over the long term

• Recent study (Financial Analysts Journal, 2004)
  → An investor holding a diversified equity portfolio has a 14% chance of a negative real return over a twenty year period
Introduction

Financial Insurance

• Derivative securities (options, futures, forwards) are tools which can be used to manage risk.
• Any investment which includes some kind of protection contains an embedded option.
• Derivative securities are used by financial institutions to hedge risk such as
  – Currency fluctuations
  – Uncertain energy costs
  – Changes in interest rates
Introduction

**Individual Investors?**

- Individual investors are often unaware that they buy/sell options
- Contracts with embedded options
  - Mortgage prepayment privileges
  - Fixed rate natural gas home heating contracts
  - Equity linked GICs
- As well, many pensions plans use derivative securities in their investment portfolio
Introduction

Do we need financial insurance?

Nortel Share Price
Example: A call option

- Suppose I have decided that I want to buy IBM stock for \( K \) in 1 month’s time
- But the price of IBM stock will fluctuate during the next month
- I don’t want to pay more than \( K \) for the stock
- But if the stock falls below \( K \), then I will be happy to pay less than \( K \) to own IBM
- Can I take out some sort of insurance to ensure that I will have to pay at most \( K \) for the stock?
A call option

• A call option gives me the right, but not the obligation to purchase the stock for a specified price (the strike price $K$) at some time in the future (the expiry date $T$).

\[ S = \text{value of stock} \]

• By purchasing a call option, at strike $K$, expiry 1 month
  – If $S > K$, I exercise the option, and buy the IBM stock for $K$.
  – If $S < K$, I let the option expire, and buy the IBM stock on the open market.
The option value

• In one month, we know for sure what the option is worth
• If $S > K$, I can buy the stock for $K$ and immediately sell it for $S$
• If $S < K$, I will not exercise the option (why pay $K$ for something worth $S$?)
• More mathematically ($T = \text{one month}$)

\[
\text{Value of call option} = V(T = 1 \text{ month}) = \max(S - K, 0)
\]

• What is a fair price for this option today?
A put option

Our previous example was for a call option

• Protection against rising prices
• Protection against falling prices can be obtained using a put option
• A put option is the right but not the obligation to sell an asset for the strike $K$ at time $T$.
• More mathematically ($t = T$)

\[
\text{Value of put option} = V(t = T) = \max(K - S, 0)
\]
**Some Jargon**

The value of the option at maturity is also called the *payoff*

\[
\begin{align*}
\text{Payoff of call} & = \max(S - K, 0) \\
\text{Payoff of put} & = \max(K - S, 0)
\end{align*}
\]

A *European Option* can only be exercised at maturity \(T\).

An *American Option* can be exercised at any time in \([0, T]\).

\[\rightsquigarrow\] The holder can decide to receive the payoff at any time.

\[\rightsquigarrow\] Most options traded on exchanges are American style.

We will consider a European call option in this example.
Simple model

A simple model: call option

• Suppose the strike price is $100, and the stock is trading today at $S_0 = 100$.

• Let’s assume a very simple model: in one month’s time, the stock price can have only two possible values

\begin{align*}
S_0 & \rightarrow S_1 = 110 \\
S_0 & \rightarrow S_2 = 90
\end{align*}
Tree model

A two state tree

What is the value of the option today? $V_0$

$V = \max(S - 100, 0)$

$S_0 = 100$

$S_1 = 110$

$S_2 = 90$

$V_0 = ?$

$V_1 = 10$

$V_2 = 0$
Tree model

Some extra information

Probability

\[ S_0 \rightarrow S_1 = \ p \]
\[ S_0 \rightarrow S_2 = (1 - p) \]

\( p = .20 \)

\( V_1 = $10 \)
\( V_2 = $0 \)
Option value?

- At this point, most people would value the option today as the discounted, expected value of the payoff

\[ V_0 = e^{-r\Delta t} (pV_1 + (1 - p)V_2) \]

\[ r = \text{interest rate} \]
\[ \Delta t = \text{One month} \]

- To keep things simple here, let’s ignore the discounting effects \((r\Delta t \approx 0)\)
Option value

• This gives (in our example: \( p = .2 \))

\[
V_0 = .2 \times $10 + .8 \times $0
\]

\[
= $2
\]

• Suppose I offer to buy this option from you for $3, will you accept my offer?

• On each option, you will make an expected profit of (price - expected payout = 3 - 2) $1
An arbitrage opportunity

• I will be very happy to buy the option from you for $3, since I will immediately exploit the arbitrage opportunity.

• I can devise a trading strategy, such that I will make a profit of $2, regardless of whether $S_0 \rightarrow 110$, or $S_0 \rightarrow 90$.

• How do I do this? Sounds like magic.
The Hedging Portfolio

- To exploit this arbitrage opportunity, I will construct a portfolio $\Pi$ which is long the option, and short $\alpha$ shares

$$\Pi = V - \alpha S$$

- A short position means I have borrowed the security, sold it, but have to give it back at some future time.

- I will choose $\alpha$ so that there is no uncertainty about the value of the portfolio at the expiry time of the option
The Hedging Portfolio

- The possible portfolio values are (in 1 month)

\[ \Pi_1 = V_1 - \alpha S_1 \]
\[ \Pi_2 = V_2 - \alpha S_2 \]

- Setting \( \Pi_1 = \Pi_2 \), and solving for \( \alpha \)

\[ \alpha = \frac{V_1 - V_2}{S_1 - S_2} = \frac{10 - 0}{110 - 90} = \frac{1}{2} \]
The Hedging Portfolio

- Today, I buy the option from you for $3.
  → I have to pay $3.
- I borrow 1/2 share of the stock, and sell it ($S_0 = \$100$). I will have to return this in one month’s time.
  → This gives me $50 in cash.
- The total value of my cash is $C_0$

\[ C_0 = \$50 - \$3 = \$47 \]
Case: $S_0 \rightarrow S_1 = 110$

Now, in one month’s time, suppose

- $S_0 \rightarrow S_1$, which means that

\[ V_1 = 10 \]
\[ -\alpha S_1 = -\frac{1}{2}(110) = -55 \]

- (I have to buy 1/2 shares at $110, and return to broker)
- So the value of the portfolio is $\Pi_1 = V_1 - \alpha S_1 = -45$.
- But I have $47 in cash, so I gain $2.
Hedge Portfolio

*Case: $S_0 \rightarrow S_2 = 90*

Similarly

- $S_0 \rightarrow S_2$, which means that

\[
V_2 = 0
\]

\[
-\alpha S_2 = -\frac{1}{2}(90) = -45
\]

- So the value of the portfolio is $\Pi_2 = V_2 - \alpha S_2 = -45$.
- But I have $47 in cash, so I gain $2.
Arbitrage opportunity

- So, no matter what happens to the stock (it can go up or down, *I don’t care*), I make a riskless profit of $2.

- What would happen in reality?
  - Arbitrageurs would buy up as many options as possible
  - This would drive up the price of the option, until the option price was the *no-arbitrage* price.

- The observed market price should be the no-arbitrage price, not the expected payoff
In our example, the no-arbitrage price is given by solving $V_0 - \alpha S_0 = \Pi_1$ for $V_0$, giving $V_0 = $5.

Note that we do not care what the probabilities of an up or down movement in the stock price are!

A bank can sell me the call option for $5(+ some profit)$, and construct the hedging portfolio.

At expiry, the portfolio can be liquidated to give exactly enough to pay off the option regardless of whether $S_0 \rightarrow S_1, S_2$ (plus a locked in profit)
More Realistic Models

• Assume more complex *stochastic models* for stock prices
• But use same basic *no-arbitrage* idea
• Black-Scholes differential equation for no-arbitrage price
  (Scholes-Merton, Nobel Prize in Economics, 1997)
  – Can be solved using numerical algorithms for no-arbitrage price
• Note that we do not care about precise path taken by stock
  (we can’t predict it)
• Only gross statistical properties (volatility)
Conclusions

- Many financial products contain embedded options
- These options are financial insurance, which are used to minimize risk
- Even though stock prices are unpredictable
- We can determine the no-arbitrage price of an option
- We can construct a hedging strategy to payout option (insurance) no matter what happens to the stock price!
- Modern finance is now a very technical discipline (Mathematics, Statistics, Computer Science)
More Reading

- Peter Bernstein, *Capital Ideas: the improbable origins of modern Wall street*
- Burton Malkiel, *A random walk down Wall Street*
- N. Taleb, *Fooled by Randomness*
- N. Taleb, *The Black Swan*
- An *Introduction to Computational Finance without Agonizing Pain* (www.scicom.uwaterloo.ca/ paforsyt/agon.pdf)
A Model for Stock Prices

• Observation: For every quoted price we see in the stock market: there is one buyer for every seller

• In each transaction
  – Buyer thinks price is going up
  – Seller thinks price is going down

• Conclusion: Stock prices follow a random walk (verified by statistical tests)
  – No observable patterns in prices
Stochastic Differential Equation for Price

- Let $S$ be the price of an underlying asset (i.e. TSX index).
- A basic model for the evolution of $S$ through time is Geometric Brownian Motion (GBM)
  \[
  \frac{dS}{S} = \mu dt + \sigma \phi \sqrt{dt}
  \]
  \[
  \mu = \text{drift rate},
  \]
  \[
  \sigma = \text{volatility},
  \]
  \[
  \phi = \text{random draw from a standard normal distribution}
  \]
Monte Carlo Paths

Geometric Brownian Motion

\[ T = 1.0 \]
\[ \sigma = 0.25 \]
\[ \mu = 0.10 \]
A lattice model

In order to price an option, don’t want to deal with the SDE directly.

We will develop a discrete lattice model of GBM.

Denote today’s stock price \((t = t^0)\) by \(S^0_0\). At \(t^1 = t^0 + \Delta t\),

\[
\begin{align*}
S^0_0 & \rightarrow S^1_1 \quad \text{; up with probability } p \\
S^0_0 & \rightarrow S^1_0 \quad \text{; down with probability } q = 1 - p
\end{align*}
\]
A lattice model

At $t^2 = t^1 + \Delta t$,

$S^1_1 \rightarrow S^2_2$; up with probability $p$

$S^1_1 \rightarrow S^2_1$; down with probability $q = 1 - p$

$S^1_0 \rightarrow S^2_0$; up with probability $p$

$S^1_0 \rightarrow S^2_1$; down with probability $q = 1 - p$
Lattice model

A Recombining Lattice

Note: $S^n_j$
Asset at timestep $n$,
node $j$.

$n$ is a superscript not a power.
At node \( j \), timestep \( t^n = n\Delta t \), asset price is denoted by \( S^n_j \).
Lattice model

Consistent with GBM

If we choose:

\[ S_{j+1}^{n+1} = S_j^n e^{\sigma \sqrt{\Delta t}} \]
\[ S_j^{n+1} = S_j^n e^{-\sigma \sqrt{\Delta t}} \]
\[ p = \frac{1}{2} \left[ 1 + \left( \frac{\mu}{\sigma} - \frac{\sigma}{2} \right) \sqrt{\Delta t} \right] \]
\[ q = 1 - p \]

then, as \( \Delta t \to 0 \), random walks on this discrete lattice converge to the solution of the GBM SDE.
Convergence to GBM

In other words, if we take many random walks on the lattice with these parameters, and record a histogram of the outcomes (an approximate probability density function).

Then, as $\Delta t \to 0$, this approximate probability density converges to the probability density function of GBM.

See notes for argument (not proof).

Idea: develop discrete no-arbitrage pricing model on the lattice, and then as $\Delta t \to 0$, this pricing model should converge to the correct solution for GBM.
Lattice model

**No-arbitrage Lattice**

We are going to use the same idea as in our simple example. At node $S^n_j$, associate an option value $V^n_j$ and a hedging portfolio $P^n_j$ as follows:

$$P^n_j = V^n_j - \alpha S^n_j$$
Lattice model

**No-arbitrage Lattice**

Value of hedging portfolio at $t = t^{n+1}$

\[
P_{j+1}^{n+1} = V_{j+1}^{n+1} - \alpha S_{j+1}^{n+1}
\]

\[
P_{j}^{n+1} = V_{j}^{n+1} - \alpha S_{j}^{n+1}
\]

Now, determine $\alpha$ so that $P_{j+1}^{n+1} = P_{j}^{n+1}$
Lattice model

No-arbitrage Lattice

\[ V_{j+1}^{n+1} - \alpha S_{j+1}^{n+1} = V_{j}^{n+1} - \alpha S_{j}^{n+1} \]  

(1)

So that

\[ \alpha = \frac{V_{j+1}^{n+1} - V_{j}^{n+1}}{S_{j+1}^{n+1} - S_{j}^{n+1}} \]  

(2)

But, this portfolio is risk free (no uncertainty about its value), so that

\[ P_{j}^{n} = e^{-r\Delta t} P_{j+1}^{n+1} \]

\[ \rightarrow V_{j}^{n} - \alpha S_{j}^{n} = e^{-r\Delta t} (V_{j+1}^{n+1} - \alpha S_{j+1}^{n+1}) \]  

(3)

Substitute (2) into (3)
No-arbitrage Lattice

\[ V_j^n = e^{-r\Delta t} \left( p^* V_{j+1}^{n+1} + (1 - p^*) V_j^{n+1} \right) \]

\[ p^* = \frac{e^{r\Delta t} - e^{-\sigma \sqrt{\Delta t}}}{e^{\sigma \sqrt{\Delta t}} - e^{-\sigma \sqrt{\Delta t}}} \] (4)

Note that the real probabilities of an up/down move do not appear in (4) (\( p^* \) does not depend on the drift \( \mu \)).

We have determined the no-arbitrage value of \( V_j^n \) in terms of \( V_{j+1}^{n+1}, V_j^{n+1} \).
No-arbitrage Lattice

Recall that the no-arbitrage value is not the expected value.

But, for $\Delta t \to 0$, then

$$0 \leq p^* \leq 1$$

so that

$$\left( p^* V_{j+1}^{n+1} + (1 - p^*) V_j^{n+1} \right)$$

looks like an expectation.

But its not the real expected value → termed the expectation in the risk neutral world.
Lattice model

**Delta Hedging**

Since our hedging portfolio is

$$P^n_j = V^n_j - \alpha S^n_j$$

$$\alpha = \frac{V^{n+1}_j - V^n_j}{S^{n+1}_j - S^n_j}$$

Note that

$$\alpha \approx \left( \frac{\partial V}{\partial S} = V_S \right)$$

$$\left( S^{n+1}_{j+1} - S^{n+1}_j \right) \to 0$$

$V_S$ is called the option delta.

This hedging strategy is called *delta hedging*
Lattice model

**Full Lattice Algorithm**

Choose $\Delta t = T/N$. Construct tree of prices

$$S^n_j = S^0_0 e^{(2j-n)\sigma \sqrt{\Delta t}}$$

$n = 0, \ldots, N$

$j = 0, \ldots, n$

We know the value of the option at $t = T = t^N$

For $j = 0, \ldots, N$

$$V^N_j = \text{Payoff}(S^N_j)$$

EndFor
Backward Recursion: European Option

\[
V^n_j = e^{-r\Delta t}(p^n V^{n+1}_{j+1} + (1 - p^n) V^{n+1}_j)
\]

For \( n = N - 1, \ldots, 0 \)

For \( j = 0, \ldots, n \)

\( V^0_0 \) is the no-arbitrage value of the option at \( t = t^0, S = S^0_0 \).

We also get an approximate value of the option delta \( = V_S \) at each node in the tree. This is the hedging parameter.
Backward Recursion: European Option

\[
V^n_j = e^{-r\Delta t} \left( p^* V^{n+1}_{j+1} + (1 - p^*) V^{n+1}_j \right)
\]

Lattice model
American Options?

Recall that an American option can be exercised at any time and the holder can receive the payoff.

So, the holder must decide, at each instant in time:

- Continue to hold the option
- Exercise immediately

A rational investor will exercise if the value of exercising is larger than the value of continuing to hold.
Lattice model

**Backward Recursion: American Option**

\[
\begin{align*}
\text{For } n &= N - 1, \ldots, 0 \\
\text{For } j &= 0, \ldots, n \\
V_j^n &:= e^{-r\Delta t}(p^*V_{j+1}^{n+1} + (1 - p^*)V_j^{n+1}) \\
V_j^n &:= \max(V_j^n, \text{Payoff}(S_j^n))
\end{align*}
\]

EndFor

EndFor

This is a dynamic programming solution to the American option optimal exercise problem.
Dynamic Programming

- Note that the optimal exercise of an American option requires solution of a global optimization problem.
- But, since we work backwards from the end state $N$, we examine all possible outcomes, and choose the optimal choice at all nodes at state $N - 1$, and so on.
- This reduces the global optimization to a set of trivial one-step optimal choices.
As $\Delta t \rightarrow 0$, the solution from the lattice algorithm converges to the solution of the Black-Scholes partial differential equation (B-S PDE)

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0$$

It can be shown that

$$(V^{lattice})^n_j = (V^{exact})^n_j + O(\Delta t)$$

$\Delta t \rightarrow 0$

$\rightarrow$ The lattice method is simply a numerical method for solving the B-S PDE.
Hedge

**Hedging**

- Let $V(S, t)$ be the value at any time of the option (computed from our lattice).

- The bank will sell the option to me for $V(S, t = 0)$ today, and construct the following portfolio $\Pi$ ($-tive \rightarrow short$)

  $$ \Pi = -V + \alpha S + B $$

  - $V =$ value of option
  - $S =$ price of underlying
  - $B =$ cash in risk free money market account
  - $\alpha =$ units of underlying

Note we have included a bank account $B$ in our total portfolio $\Pi$. 
Hedging

So, what does the bank do?

- Sell the option today for \( V(S, t = 0) \) (lattice price).
- Construct the portfolio \( \Pi \), by buying \( \alpha(S, t = 0) \) units at price \( S \), and depositing \( B \) in the money market account.
- As \( t \to t + \Delta t \), \( S \to S + \Delta S \), bank rebalances the hedge, by buying/selling underlying so that \( \alpha(S + \Delta S, t + \Delta t) = V_S \).
- Hedging portfolio is \textit{Delta Neutral}. 

Hedge
Delta Hedge

Delta Hedging

- This strategy is called *Delta Hedging*
- Note that this is a dynamic strategy (rebalanced at finite intervals)
- It is self-financing, i.e. once the bank collects cash from selling option, no further injection of cash into Π is required.
- At time $T$ in the future, the bank liquidates Π, pays off short option position, at zero gain/loss, *regardless of random path followed by $S$*. 
No-arbitrage

No-arbitrage Price

• The value of the option $V(S, t)$ is the no-arbitrage value
• $V(S, t = 0)$ is the cost of setting up the portfolio $\Pi$ at $t = 0$
• The value of the option is not the discounted expected payoff

Does this actually work? Can we construct a hedge so we can’t lose, regardless of the random path followed by $S$?
• Simulate a random price path, along path, carry out delta hedge at finite rebalancing times (not a perfect hedge)
• Liquidate portfolio at expiry, pay off option holder, record profit and loss
Monte Carlo Delta Hedge Simulation: Discounted Relative Profit and Loss

Rebalance Weekly

T = 1 year
Vol = .25
r = .05
Put
40,000 simulations

Probability Density: Normalized Profit and Loss

Rebalance Twice Daily

T = 1 year
Vol = .25
r = .05
Put
40,000 simulations

Probability Density: Normalized Profit and Loss
Real World

**Reality**

- Nobody hedges at infinitesimal intervals, volatility $\neq const.$, GBM not a perfect model
- Bank wants to make a profit

\[
\begin{align*}
V_{\text{buy}} &= V(S, t)^{\text{model}} + \epsilon_1 + \epsilon_2 \\
V_{\text{sell}} &= V(S, t)^{\text{model}} - \epsilon_1 - \epsilon_2 \\
\epsilon_1 &= \text{profit} \quad \epsilon_2 = \text{compensation for imperfect hedge} \\
V_{\text{buy}} - V_{\text{sell}} &= \text{bid-ask spread}
\end{align*}
\]
What’s Wrong with GBM?

• Equity return data suggests market has *jumps* in addition to GBM
  – Sudden discontinuous changes in price
GBM?

What’s Wrong with GBM?

- Volatility not constant
- VIX index is a measure of instantaneous volatility ($S&P500$)
- Volatility is itself stochastic
Research Challenges

• Pricing and hedging options under jump processes and stochastic volatility (Monte Carlo, PDE methods)
• Pricing exotic options (Numerical soln of PDEs)
• Optimal trade execution (algorithmic trading)
  – Optimal stochastic control
• Model calibration (optimization)
Notes Reading

An Introduction to Computational Finance without Agonizing Pain (www.scicom.uwaterloo.ca/ paforsyt/agon.pdf)

Sections: 1, 2.1, 2.2, 2.3, 2.4, 5

If you have time: 2.5, 2.6, 8.1, 8.2

And more if you like!

CS476, Winter 2008 Introduction to numeric computation for financial modelling